Bochner's Theorem

Theorem 0.1 (Bochner)**.** *If $\phi \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ is positive definite, then $\phi(x) = \int e^{2\pi i x t} d\mu(t)$ for all $x \in \mathbb{R}$ and one finite measure μ on $\mathbb{R}$.*

We need two lemmas.

Lemma 0.2. *If $f \in L'(\mathbb{R})$, $\phi \in C(\mathbb{R})$ is positive definite, then $\int f * \tilde{f} \phi dx \geq 0$ where $\tilde{f}(x) = \overline{f(-x)}$.*

The proof of the lemma is left to the reader as an exercise.

Lemma 0.3. *If $f \in \mathcal{S}(\mathbb{R})$ and $\chi \in \mathcal{S}(\mathbb{R})$, then*

$$\chi(\frac{x}{n}) f(x) \to_{n \to \infty} \chi(0) f(x) \text{ in } \mathcal{S}.$$

Lemma 0.3 was discussed in class and we will not prove it here.

Proof of Bochner's theorem. We know ϕ and would like to find a finite μ such that

$$\phi(x) = \int e^{2\pi i x t} d\mu(t).$$

We would like to define a positive functional on $C_c(\mathbb{R})$ which gives a measure by Riesz representation theorem.

We first define the functional on $\mathcal{S}$. For $f \in \mathcal{S}$ we define

$$\Lambda(f) = \int \check{f}(x) \phi(x) dx.$$

$f \in \mathcal{S}$ implies that $\check{f} \in \mathcal{S} \subset L'$. Also, since $\phi \in L^\infty$ we have $|\Lambda(f)| \leq \|f\|_1 \|\phi\|_\infty < \infty$. Thus, Λ is well-defined on $\mathcal{S}$.

Now, we want to obtain positivity on C_c^∞. Let $f \in C_c^\infty(\mathbb{R})$ with $f \geq 0$ and let $\epsilon > 0$ be given. Define $F = \sqrt{f + \epsilon e^{-\pi x^2}}$. We have $F > 0$ and it is smooth, in fact $F \in \mathcal{S}$ since $f \in \mathcal{S}$. Applying Lemma 0.2 with $f = \check{F}$ we obtain $\int \check{F} * \tilde{\check{F}} \phi dx \geq 0$. However, $\check{F^2} = \check{F} * \tilde{\check{F}}$. Hence,

$$\Lambda(F^2) = \int \check{F^2} \phi dx = \int \check{F} * \tilde{\check{F}} \phi dx \geq 0.$$

So, $\Lambda(F^2) = \Lambda(f) + \epsilon \Lambda(e^{-\pi x^2}) \geq 0$. Since $\epsilon > 0$ is arbitrary, we obtain $\Lambda(f) \geq 0$ for any $f \in C_c^\infty$.

On the other hand, C_c^∞ is dense in C_c in the uniform norm (cf. Folland,Theorem(8.14) (c)). Thus, Λ extends linearly to C_c. Let $f \in C_c$ be given with $f \geq 0$ and let $f_n \in C_c^\infty$ with $\|f_n - f\|_\infty \to 0$. Assume that $supp(f_n) \subset K$ for all n. Pick a nonnegative function $g \in C_c^\infty$

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such that $g = 1$ on K. Then, $gf_n = f_n \geq -g\|f_n - f\|_\infty$ and we have $\Lambda(f_n) \geq -\Lambda(g)\|f_n - f\|_\infty$. Letting $n \to \infty$ we obtain $\Lambda(f) \geq 0$. Hence, we obtain a positive functional on $C_c(\mathbb{R})$. So, by Riesz representation theorem there exists a unique Radon measure μ such that $\Lambda(f) = \int f d\mu$ for all $f \in C_c$. On the other hand, we know that $\Lambda(f) = \int \check{f}\phi dx$ for all $f \in \mathcal{S}$.

For any $f \in C_c^\infty$ and any $\epsilon > 0$ we can choose $n > 0$ sufficiently big so that

$$e^{-\pi t^2/n^2}(\|f\|_\infty + \epsilon) \geq f(t) \geq -(\|f\|_\infty + \epsilon)e^{-\pi t^2/n^2}.$$

Recall that $\widehat{e^{-\pi t^2/n^2}} = ne^{-\pi n^2 x^2}$. Applying Λ we obtain

$$\|\varphi\|_\infty(\|f\|_\infty + \epsilon)\int ne^{-\pi n^2 x^2}dx \geq \Lambda(f) \geq -(\|f\|_\infty + \epsilon)\|\varphi\|_\infty \int ne^{-\pi n^2 x^2}dx,$$

which implies $|\Lambda(f)| \leq \|f\|_\infty \|\varphi\|_\infty$ since $\int ne^{-\pi n^2 x^2}dx = 1$ and $\epsilon > 0$ is arbitrary. Thus, Λ is a bounded functional on C_c since C_c^∞ is dense in C_c. In particular, this gives that μ is a finite measure.

On the other hand, using Lemma 0.3 we obtain the formula

$$\int \check{f}\phi dx = \int f d\mu \text{ for any } f \in \mathcal{S}.$$

For $f_n(t) = e^{2\pi i y t - \pi t^2/n^2} \in \mathcal{S}$ we have $\check{f}_n(x) = ne^{-\pi n^2(x+y)^2}$. Hence,

$$\Lambda(f_n) = \int ne^{-\pi n^2(x+y)^2}\varphi(x)dx = \int e^{2\pi i y t - \pi t^2/n^2}d\mu(t).$$

We know that $\int ne^{-\pi n^2(x+y)^2}\varphi(x)dx \to \phi(y)$ as $n \to \infty$ (see Folland's book Theorem 8.14 (c)). Also, from Dominated Convergence theorem $\int e^{2\pi i y t - \pi t^2/n^2}d\mu(t) \to \int e^{2\pi i y t}dt$ as $n \to \infty$. Thus,

$$\phi(y) = \int e^{2\pi i y t}dt.$$

□