17/05/2010 Mon

Distribution:

$F \in \mathcal{D}'(O)$ is a functional $\varphi \in C_c^\infty(O) \longmapsto \langle F, \varphi \rangle$

$O \subset \mathbb{R}^d$

Tempered distribution:

$F \in \mathcal{S}'(\mathbb{R}^d)$ is a functional $\varphi \in \mathcal{S}(\mathbb{R}^d) \mapsto \langle F, \varphi \rangle$.

Note $\mathcal{S}'(\mathbb{R}^d) \subseteq \mathcal{D}'(\mathbb{R}^d)$

$\mathcal{D}'(O_1) \longrightarrow \mathcal{D}'(O_2)$ if $O_2 \subseteq O_1$

restriction map.

If $F \in \mathcal{D}'(O)$ or $F \in \mathcal{S}'$ then $\langle \frac{\partial}{\partial x_i} F, \varphi \rangle := - \langle F, \frac{\partial}{\partial x_i} \varphi \rangle$	Definition.

If $F \in C^\infty(O)$ then $F \in L^1_{loc}$ ← so gives a distribution &

$\frac{\partial}{\partial x_i} F$. (in the sense of distr) $=$ L^1_{loc} - function $\frac{\partial}{\partial x_i} F$

(defined as in Analysis).

We now define the <u>Fourier-transform of tempered distributions</u>

<u>Recall</u> $\widehat{\mathcal{S}} = \mathcal{S}$, $\int f \hat{g} \, dx = \int \hat{f} g \, dt$

<u>Def</u>: If $F \in \mathcal{S}'$ then we define $\widehat{F} \in \mathcal{S}'$ by

$$\langle \widehat{F}, \varphi \rangle = \langle F, \widehat{\varphi} \rangle$$

$$\varphi \in \mathcal{S}$$

<u>Rk</u>: If $F \in L^1$ then we already defined $\widehat{F}$ but we know $\int F \widehat{\varphi} = \int \widehat{F} \varphi$.

← defined as before

$\int F \hat{\varphi} = \langle F, \hat{\varphi} \rangle$

$\int \hat{F} \varphi = \langle \hat{F}, \varphi \rangle$

If $F \in L^2$ then we already defined $\hat{F} \in L^2$ &

$$\langle F, \hat{\varphi} \rangle_{\text{distribution}} = \int F \hat{\varphi} \, d\underline{x} = \langle F, \bar{\hat{\varphi}} \rangle_{L^2} = \langle \hat{F}, \hat{\bar{\hat{\varphi}}} \rangle_{L^2}$$

$$= \int \hat{F} \bar{\hat{\bar{\hat{\varphi}}}} = \langle \hat{F}, \varphi \rangle_{\text{dist.}}$$

$\therefore \quad \hat{F}^{\text{distr}} = \hat{F}^{L^2}$

$\hat{F}$ is again continuous b/c $\varphi_n \to \varphi$ in S

$\Rightarrow \hat{\varphi}_n \to \hat{\varphi}$ in S

$\Rightarrow \langle \hat{F}, \varphi_n \rangle = \langle F, \hat{\varphi}_n \rangle \to \langle F, \hat{\varphi} \rangle = \langle \hat{F}, \varphi \rangle$

12. Sobolev spaces, and elliptic regularity, approach via S'.

(Folland, Real Analysis)

Def: $H_k(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}) :$ f has L^2-partial derivatives for all $\underline{\alpha} \in \mathbb{N}_0^d$ with $|\underline{\alpha}| \leq k \}$.

$f_j \in L^2$ is an L^2-der. for $\frac{\partial}{\partial x_j}$ if $\int f_j \varphi = - \langle f, \frac{\partial}{\partial x_j} \varphi \rangle$, $\varphi \in C_c^\infty(\mathbb{R}^d)$

Which norm should we use?

Maybe $\sqrt{\sum_{|\alpha| \leq k} \| \frac{\partial}{\partial \underline{x}^{\underline{\alpha}}} f \|_2^2}$

We will use

$$\| (1 + \|\underline{t}\|^2)^{k/2} \hat{f}(\underline{t}) \|_{L^2} = \|f\|_{H_k}$$

Lemma: If $F \in S' \Rightarrow \widehat{\frac{\partial}{\partial x_j} F} = 2\pi i t_j \hat{F}$

Pf: $\langle \widehat{\frac{\partial F}{\partial x_j}}, \varphi \rangle = \langle \frac{\partial}{\partial x_j} F, \hat{\varphi} \rangle = - \langle F, \frac{\partial}{\partial x_j} \hat{\varphi} \rangle$

$$= -\langle F, -\widehat{2\pi i t_j \varphi}\rangle = \langle \hat{F}, 2\pi i t_j \varphi\rangle \underset{\text{def.}}{\overset{\uparrow}{=}} \langle 2\pi i t_j \hat{F}, \varphi\rangle$$ □

If $f \in H_k^{(k \geq 1)}$ as defined above, then $\frac{\partial}{\partial x_j} f$ exists in L^2.
Hence this L^1_{loc} - fn is the
tempered distr. defined by $\frac{\partial}{\partial x_j} f \Rightarrow$ by the lemma
$$\widehat{\frac{\partial}{\partial x_j} f} = 2\pi i t_j \hat{f} \in L^2.$$

$$1 + \sum |t_j| \asymp (1 + \|t\|^2)^{1/2}$$

$$\sum_{|\alpha| \leq k} |t^\alpha| \asymp (1 + \|t\|^2)^{k/2}$$

<u>Def</u>: Let $s \in \mathbb{R}$ then we define
$$H_s(\mathbb{R}^d) = \{ f \in S' \mid (1+\|t\|^2)^{s/2} \hat{f} \in L^2, \hat{f} \in L^1_{loc} \}$$
$$\|f\|_{H_s} = \|(1+\|t\|^2)^{s/2} \hat{f}\|_{L^2}$$

If $s \geq 0$ then
$$H_s(\mathbb{R}^d) \longrightarrow H_0(\mathbb{R}^d) = L^2(\mathbb{R}^d) \quad \text{cts}$$
$$\|f\|_2 = \|\hat{f}\|_{L^2} \leq \|(1+\|t\|^2)^{s/2} \hat{f}\|_{L^2} = \|f\|_{H_s}.$$

exc: for which s, $\delta_0 \in H_s$.

If $s \geq k \in \mathbb{N}_0$, $H_s(\mathbb{R}^d) \longrightarrow H_k(\mathbb{R}^d)$ & $\|f\|_{H_k} \leq \|f\|_{H_s}$.

<u>sobolev embedding thm</u>:

<u>Thm</u>: Suppose $s > k + \frac{1}{2} d$. Then

a) $f \in H_s$ then $\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} f\right)^\wedge \in L^1$, $|\alpha| \leq k$.

and $\|(\frac{\partial^{|\alpha|}}{\partial x^\alpha} f)^\wedge\|_{L^1} \leq C_{k,s,d} \|f\|_{H_s}$

b) $H_s \subseteq C_0^k$ & continuity of inclusion.

Pf:

$$(2\pi)^{-|\alpha|}\int |(\frac{\partial^{|\alpha|}}{\partial \underline{x}^\alpha} f)^\wedge|\,dt = \int |\underline{t}^\alpha \hat{f}|\,dt \leq \left(\int (1+\|t\|^2)^s |\hat{f}(t)|^2 dt\right)^{1/2} \cdot$$

$$\cdot\left(\int \frac{|\underline{t}^\alpha|^2}{(1+\|t\|^2)^s}\,dt\right)^{1/2}.$$

$\uparrow$ $\|f\|_{H_s} < \infty$

$s \geq 0 \Rightarrow f \in L^2, \hat{f} \in L^2$

$s \geq k \Rightarrow \frac{\partial}{\partial x^\alpha} f \in L^2$ for $|\alpha| \leq k$

$$\left(\int \frac{|\underline{t}^\alpha|^2}{(1+\|t\|^2)^s}\,dt\right)^{1/2} \ll \left(\int \frac{1}{(1+\|t\|^2)^{s-k}}\right)^{1/2} < \infty \quad \text{since } 2(s-k) > d.$$

Since $s > k + \frac{1}{2}d$, $k \geq 0$. we have

$$\|\hat{f}\|_{L^1} < \infty \quad \Rightarrow \quad f \in C_0(\mathbb{R}^d) \quad \text{a.e.}$$

Also, $\|t_j \hat{f}\|_{L^1} < \infty$ (if $k \geq 1$)

$\underset{\text{by Prop §10}}{\Longrightarrow}$ $\frac{\partial}{\partial x_j} f$ exists & $\in C_0(\mathbb{R}^d)$.

20/05/2010 Thu

Recall: If $\underline{x}^\alpha \varphi \in L^1(\mathbb{R}^d)$ for $|\alpha| \leq k$ then

$$\hat{\varphi} \in C_0^k, \quad \frac{\partial^{|\alpha|}}{\partial \underline{t}^\alpha}(\hat{\varphi}) = [(-2\pi i \underline{x})^\alpha \varphi]^\wedge, \quad |\alpha| \leq k$$

Recall: for $s \in \mathbb{R}$

$$H_s(\mathbb{R}^d) = \{ f \in \mathcal{S}' \mid \hat{f} \in L^1_{loc}, (1+|t|^2)^{s/2} \hat{f}(t) \in L^2 \}$$

$$\|f\|_s = \|(1+\|t\|^2)^{s/2} \hat{f}\|_{L^2}$$

Lemma. $s \geq k \geq 1$

$\Rightarrow f \in H_s$ implies f has L^2-part. der. up to order k.

In fact,
$$\left(\frac{\partial}{\partial x_j} f\right)^\wedge = 2\pi i t_j \hat{f}(t)$$

Pf: Define $f_j = (2\pi i t_j \hat{f})^\vee \in L^2$ by assumption. $s \geq 1$.

$\langle f_j, \varphi \rangle = \langle 2\pi i t_j \hat{f}, \varphi^\vee \rangle$ $\boxed{\text{since } \int \check{F} \varphi = \int F \check{\varphi}}$

$\varphi \in \mathcal{S}$

$$= \langle \hat{f}, 2\pi i t_j \varphi^\vee \rangle = \langle f, -(-2\pi i t_j \varphi^\vee)^\wedge \rangle$$
$$= \langle f, -\frac{\partial}{\partial x_j} (\varphi^\vee)^\wedge \rangle = -\langle f, \frac{\partial}{\partial x_j} \varphi \rangle$$

$\square$

[$f \in H_s(\mathbb{R}^d) \Rightarrow \frac{\partial}{\partial x_j} f \in H_{s-1}(\mathbb{R}^d)$.

Lemma: If $F \in \mathcal{S}'$

$$\Rightarrow \left(\frac{\partial}{\partial x_j} F\right)^\wedge = 2\pi i t_j \hat{F}$$ $\square$

Thm (Sobolev Embedding Thm).

If $s > k + \frac{1}{2} d$ then $H_s \hookrightarrow C_0^k$ and the inclusion map is continuous.

Pf: $k = 0$ we only demand $s > \frac{1}{2} d$ $\left(\Leftrightarrow \int \frac{1}{(1+\|t\|^2)^s} dt < \infty\right)$

$f \in H_s \Rightarrow (1 + \|t\|^2)^{s/2} \hat{f} \in L^2$

$\Rightarrow \hat{f} \in L^1 \Rightarrow \underset{\text{in } \mathcal{S}'}{f = } (\hat{f})^\vee \in C_0$

Cor: If $f \in H_s(\mathbb{R}^d)$ $\forall s \in \mathbb{R}$ (or $s \in \mathbb{N}$) then $f \in C_0^\infty(\mathbb{R}^d)$

Thm Elliptic regularity for Δ on $\mathbb{R}^d$

If $F \in \mathcal{S}' \cap H_\sigma$ and $\Delta F \in H_s$, then $F \in H_{s+2}$

[Note: $\langle \Delta F, \varphi \rangle = \langle F, \Delta \varphi \rangle$]

Pf: $F \in \mathcal{S}' \cap H_\sigma \Rightarrow (1+\|t\|^2)^{\sigma/2} \hat{F} \in L^2 \qquad (*)$

$$\Delta F \in H_s \Rightarrow \underbrace{(1+\|t\|^2)^{s/2} \left(\sum_{j=1}^d t_j^2 \right) \hat{F}(\underline{t})}_{\text{roughly } (1+\|t\|^2)^{\frac{s+2}{2}} \hat{F}(\underline{t})} \in L^2$$

$$\Downarrow$$

$$(1+\|t\|^2)^{\frac{s+2}{2}} \hat{F}(\underline{t})\Big|_{\mathbb{R}^d \setminus B_1^{\mathbb{R}^d}(0)} \in L^2$$

$$(*) \Rightarrow \hat{F}\Big|_{B_1^{\mathbb{R}^d}(0)} \in L^2$$

$$\Rightarrow (1+\|t\|^2)^{\frac{s+2}{2}} \hat{F} \in L^2$$

$\square$

Def: $P = \sum_{|\underline{\alpha}| \le m} a_{\underline{\alpha}} \cdot \frac{\partial^{|\alpha|}}{\partial \underline{x}^{\underline{\alpha}}}$ with $a_{\underline{\alpha}} \in \mathbb{C}$ is a constant coeff. partial diff. operator of degree $\le m$. It is degree m if it is degree $\le m$ but not degree $\le m-1$.

Def: A partial diff. oper. is called elliptic of order m if it is a partial diff. oper. of order m and

$P_m(\underline{T}) = \sum_{|\underline{\alpha}| = m} a_{\underline{\alpha}} \underline{T}^{\underline{\alpha}} \in \mathbb{C}[T_1, \dots, T_d]$ satisfies that

$P_m(\underline{t}) \neq 0$ for all $\underline{t} \in \mathbb{R}^d \setminus \{0\}$.

Δ is an elliptic p.d.op.

$$\Delta = \frac{\partial^2}{\partial x^2} + \dots + \frac{\partial^2}{\partial x_d^2}, \quad m = 2$$

$P_2(T_1, \dots, T_d) = T_1^2 + \dots + T_d^2$

The same theorem generalizes to elliptic p.d.o.

We want to prove $\left.\begin{array}{l} u \in H_0(O) \\ \Delta u \in H_s(O) \end{array}\right\} \Rightarrow u \in H_{s+2}(O)$, $O \subseteq \mathbb{R}^d$ open

<u>Recall:</u> $F \in \mathcal{D}'(O)$, $\varphi \in C_c^\infty(O) \Rightarrow F \cdot \varphi \in \mathcal{D}'(O)$

Actually: $F \cdot \varphi \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle F \cdot \varphi, \psi \rangle = \langle F, \underbrace{\varphi \cdot \psi}_{\in C_c^\infty(O)} \rangle$$

Continuity: $\varphi_n \to \varphi$ in $\mathcal{S}$

$\Rightarrow \underset{\in C_c^\infty(O)}{\psi \varphi_n} \to \underset{\in C_c^\infty(O)}{\psi \varphi}$ $\qquad$ $\mathrm{supp}(\psi \varphi_n) \subseteq \mathrm{supp}(\psi) = K \subseteq O$

$$\begin{array}{l} \langle F, \psi \varphi_n \rangle \to \langle F, \psi, \varphi \rangle = \langle F\psi, \varphi \rangle \\ \quad \| \\ \langle F\varphi, \varphi_n \rangle. \end{array}$$

<u>Def:</u> We say $f \in \mathcal{D}'(O)$ is in $H_s^{loc}(O)$ if $\psi \cdot f \in H_s(\mathbb{R}^d)$ for any $\psi \in C_c^\infty(O)$.

E.g. $C \subseteq H_0^{loc}$

$C^k \subseteq H_k^{loc}$

Thm: Elliptic regularity for Δ on $O \subseteq \mathbb{R}^d$.

Suppose $O \subseteq \mathbb{R}^d$ open & $u \in \mathcal{D}'(O)$ satisfies

(σ) $u \in H_\sigma^{loc}(O)$ for some $\sigma \in \mathbb{R}$ [not necessary]

(s) $\Delta u \in H_s^{loc}(O)$ for some $s \in \mathbb{R}$

then actually $u \in H_{s+2}^{loc}(O)$

In particular, if $u \in H_\sigma^{loc}(O)$ and

or $\begin{cases} \cdot\ \Delta u \in C^\infty(O) \\ \cdot\ \Delta u = \lambda u \\ \cdot\ \Delta^n u \in H_s^{loc}(O) \text{ for all } n \end{cases}$

then $u \in C^\infty(O)$.

Pf: $\psi \in C_c^\infty(O)$

claim: $\Delta(\psi \cdot u) = (\Delta\psi)\cdot u + \psi(\Delta u) + 2\sum_j \left(\frac{\partial}{\partial x_j}\psi\right)\left(\frac{\partial}{\partial x_j}u\right)$

- $\psi \cdot u \in H_\sigma$ by (σ) ✓
- $(\Delta\psi)u \in H_\sigma$ by (σ) ✓
- $\psi(\Delta u) \in H_s$ by (s) ✓
- $2\sum_j \left(\frac{\partial}{\partial x_j}\psi\right)\left(\frac{\partial}{\partial x_j}u\right) \in H_{\sigma-1}$ (need to check!)

$\Rightarrow \psi \cdot u \in H_{\min(\sigma-1, s)+2}$

↑ by Thm on $\mathbb{R}^d$.

$\Rightarrow \psi \cdot u \in H_{\min(\sigma+1, s+2)}$. $\forall \psi \in C_c^\infty \Rightarrow u \in H^{loc}_{\min(\sigma+1, s+2)}$

If $\sigma+1 < s+2$ then may replace σ by $\sigma+1$ w/o affecting the assumptions.

Iterate the argument until $\min(\sigma+k, s+2) = s+2$

Left to check:

- Equality in the claim
- $\sum \left(\frac{\partial}{\partial x_j}\psi\right)\left(\frac{\partial}{\partial x_j}u\right) \in H_{\sigma+1}$.