8. Locally convex spaces: uniform, strong, weak and weak* topologies

[see Folland, Chapter 5.4 Real Analysis]

<u>Def.</u> A <u>topological vector space</u> is a vector space X over $\mathbb{R}$ or $\mathbb{C}$ equipped with a topology for which

$$(x,y) \mapsto x+y$$

$$(\lambda, x) \mapsto \lambda x$$

are continuous functions.

<u>Def</u> A top. vector space is <u>locally convex</u> if $0 \in X$ has a basis of neighborhoods consisting of convex sets

Collections of semi-norms define locally convex topologies:

Prop. Let X be a vector space. Let $\{p_\alpha : \alpha \in A\}$ be a family of semi-norms on X. Then the topology generated by the sets

$$U_{x,\alpha,\varepsilon} = \{y \in X \mid p_\alpha(y-x) < \varepsilon\}$$

makes X into a locally convex space. Moreover, for any $x \in X$ finite intersecti[ons] of sets of the form $U_{x,\alpha,\varepsilon}$ give a basis of the neighborhoods of x.

Pf. By def. the topology consists of all unions of finite intersections of sets of the form $U_{x\alpha,\varepsilon}$.

Suppose $x \in \bigcap_{i=1}^{k} U_{x_i,\alpha_i,\varepsilon_i}$. Define $\delta_i = \varepsilon_i - p_{\alpha_i}(x$
Then $U_{x,\alpha_i,\delta_i} \subseteq U_{x_i,\alpha_i,\varepsilon_i}$ by the triangle ineq. for α_i. This shows $\bigcap_{i=1}^{k} U_{x,\alpha_i,\delta_i} \subseteq \bigcap_{i=1}^{k} U_{x_i,\alpha_i,\varepsilon_i}$

and so the last claim of the prop.

To prove continuity note first that

$$U_{x,\alpha,\frac{\varepsilon}{2}} + U_{y,\alpha,\frac{\varepsilon}{2}} \subseteq U_{x+y,\alpha,\varepsilon}$$

which quickly implies continuity of $+$.

Second note that

$$p_\alpha(\eta y - \lambda x) \leq p_\alpha(\eta(y-x)) + p_\alpha((\eta-\lambda)x)$$

$$\leq |\eta|\, p_\alpha(y-x) + |\eta-\lambda|\, p_\alpha(x)$$

which implies

$$\left\{ \eta \in \mathbb{R}_{\mathbb{C}} \;\middle|\; \begin{matrix} |\eta-\lambda| < \frac{\varepsilon}{2p_\alpha(x)} \\ |\eta| < |\lambda|+1 \end{matrix} \right\} \cdot U_{x,\alpha,\frac{\varepsilon}{2(|\lambda|+1)}} \subseteq U_{\lambda x,\alpha,\varepsilon}$$

and so continuity of $\cdot$.

$\square$

Examples

of seminorms

1 • Banach spaces

• $C(\mathbb{R}^*)$ with the topology of uniform convergence on compact sets

ω

$$p_N(f) = \| f|_{[-N,N]} \|_\infty$$

• $L^1_{loc}(\mathbb{R}^n) = \{ [f] : f : \mathbb{R}^n \to \mathbb{C}$ measurable and integrable on every compact subset $\}$

ω

$$p_N(f) = \int_{B_N(0)} |f(\underline{x})| \, d\underline{x}$$

• $C^\infty([0,1])$

ω

$$p_N(f) = \| f^{(N)} \|_\infty$$

• $C_c(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C}$ cont. with compact support,

$\cancel{\omega}$

$p_F(f) = \| f F \|_\infty$ for any $F \in C(\mathbb{R}$

[Why not just $\|f\|_\infty$?]

- X a Banach space and X^* its dual.

The <u>weak topology</u> on X is the topology induced by the seminorms on X

$$p_\ell : x \mapsto |\ell(x)| \quad \text{for } \ell \in X^*.$$

- X, X^* as above.

The weak* topology on X^* is the topology induced by the seminorms on X^*:

$$p_x : \ell \mapsto |\ell(x)| \quad \text{for } x \in X.$$

<u>Note</u>: On X^* there is also a weak topology, which may not be equal to the weak* topology.

<u>Note</u> The weak* topology is the "topology of pointwise convergence" when elements of X^* are considered functions on X.

<u>Theorem</u> Tychonoff-Alaoglu

Let X be a Banach space. Then the closed unit ball

$$B^* = \{ \ell \in X^* \mid \|\ell\| \leq 1 \}$$

is compact in the weak* topology.

<u>Pf.</u> $$B^* \subseteq \prod_{x \in X} \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \|x\| \} = P$$

in the sense that

$$\ell \longmapsto (\ell(x) : x \in X)$$

The weak* topology on X^* restricted to B^* is the product topology in P.

One only has to show that B^* is closed in P: P consists of all maps, and B^* of all linear maps (continuity of the linear maps is garanteed by def. of P).

Suppose $f \in P$ is not linear, then one of two things happen: either there exists $x, y \in X$ with $f(x+y) \neq f(x) + f(y)$ or there exists x, λ with $f(\lambda x) \neq \lambda f(x)$. Suppose the former holds (the second case is similar): let $\varepsilon = \frac{\| f(x+y) - f(x) - f(y) \|}{3}$ and let

$g \in P$ belong to the neighborhood of f defined by

$$|g(x+y) - f(x+y)| < \varepsilon$$
$$|g(x) - f(x)| < \varepsilon$$
$$|g(y) - f(y)| < \varepsilon$$

then $g(x+y) \neq g(x) + g(y)$ which shows that g is also not in P.

Therefore, the image of B^* in P is closed & so compact by Tychonoff's thm. $\square$

- Another important example:

Let X, Y be Banach spaces and let $B(X,Y)$ be the Banach space of all bounded linear operators from X to Y. On $B(X,Y)$ there are three topologies:

I) The uniform topology defined by the norm on $B(X,Y)$.

II) The strong operator topology defined by the seminorm

$$p_x(T) = \|Tx\|_Y \quad \text{for all } x \in X$$

III The weak operator topology defined by the seminorms

$$p_{x,\ell}(T) = |\ell(Tx)| \quad \text{for all } x \in X \text{ and } \ell \in Y$$

Why is it not enough to consider only the uniform topology?

<u>Example</u> $H = L^2(\mathbb{R})$

$$U_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$f(x) \longmapsto f(x+t)$$

is a unitary operator for every $t \in \mathbb{R}$.

Hence we have a map

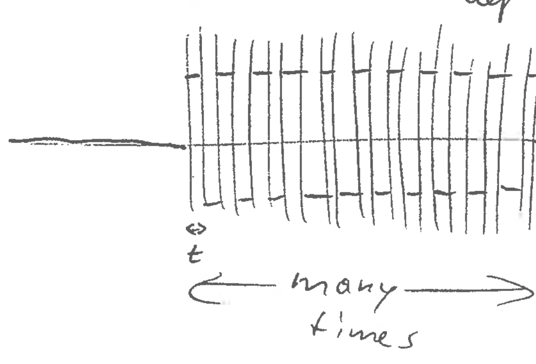
$$t \in \mathbb{R} \longrightarrow \mathcal{U}(L^2(\mathbb{R}), L^2(\mathbb{R}))$$

which is a <u>unitary representation</u> of $\mathbb{R}$. Is this map continuous?

in the uniform topology?

No: if $t \neq 0$, then $\|U_t - Id\| = 2$

defined by a supremum

$f =$ 

height s.t. $\|f\|_2 = 1$

then $U_t f$ is almos $-f$

In the strong operator topology?

Yes: if $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$ then there exists some $F \in C_c(\mathbb{R})$ with

$$\|F - f\|_2 < \varepsilon.$$

$t \mapsto U_t F \in L^2(\mathbb{R})$ is continuous

$t \mapsto U_t f$ is continuous [up to 2ε for any ε].

Remark: If a LC vector space is defined by countably many seminorms $\{p_1, -, p_n, --\}$ then the topology is metrizable:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

Def. A Frechet space is a LC vector space X defined by countably many seminorms such that X is complete w.r.t. the induced metric $d(\cdot,\cdot)$ as above.

<u>Examples</u> • $C^\infty([a,b])$

• $SubPol(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \mid f$ continuous and for any $n \geq 0$ $\|x^n f(x)\|_\infty < \infty \}$

seminorms

$$\|f\|_n := \|x^n f(x)\|_\infty$$

• $\mathcal{S}(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C} \mid f \in C^\infty(\mathbb{R}^d) \cap SubPol(\mathbb{R}^d)$ s.t. all partial derivatives also belong to $SubPol(\mathbb{R}^d) \}$

Schwarz-space

seminorms $\underline{\alpha} \in \mathbb{N}_0^d$ "a multi-index" $|\underline{\alpha}| = \alpha_1 + \dots + \alpha_d$

$$\|f\|_{\alpha,n} = \left\| (1+|x|^2)^{n/2} \frac{\partial^{|\alpha|}}{\partial x_{\underline{\alpha}}} f \right\|_\infty < \infty$$

by def of spa

These are <u>Frechet spaces.</u>

[which we will not prove here, but you may prove it directly] next

- Fourier series & Bochner's theorem

- Fourier transform