9. Fourier series & Bochner's thm

Let $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \underset{\uparrow}{\cong} [0,1)^d$
as a measure space

The characters $e_{\underline{n}}$, $\underline{n} \in \mathbb{Z}^d$ are defined by

$$e_{\underline{n}}(\underline{x}) = e^{2\pi i (\underline{n} \cdot \underline{x})} = e^{2\pi i (n_1 x_1 + \dots + n_d x_d)}.$$

The characters are <u>orthogonal to each other</u>:

$$\langle e_{\underline{n}}, e_{\underline{m}} \rangle = \int_{\mathbb{T}^d} e_{\underline{n}}(\underline{x}) \overline{e_{\underline{m}}(\underline{x})} \, d\underline{x}$$

$$= \int_{\mathbb{T}^d} e^{2\pi i (\underline{n} \cdot \underline{x} - \underline{m} \cdot \underline{x})} dx$$

$$= \int e_{\underline{n}-\underline{m}}(\underline{x}) \, dx$$

If $\underline{n} = \underline{m}$ then $\|e_n\| = 1$.

If $\underline{n} \neq \underline{m}$ then for some $\underline{y} \in \mathbb{T}^d$
$e_{\underline{n}-\underline{m}}(\underline{y}) \neq 1$

and so $\displaystyle\int_{\mathbb{T}^d} e_{\underline{n}-\underline{m}}(\underline{x})\,d\underline{x} =$

$$\int_{\mathbb{T}^d} e_{\underline{n}-\underline{m}}(\underline{x}+\underline{y})\,d\underline{x} =$$

$$\underbrace{e_{\underline{n}-\underline{m}}(\underline{y})}_{\neq 1} \int_{\mathbb{T}^d} e_{\underline{n}-\underline{m}}(\underline{x})\,dx$$

which shows $\displaystyle\int_{\mathbb{T}^d} e_{\underline{n}-\underline{m}}(\underline{x})\,dx = 0$

The characters form a <u>complete</u> <u>orthonormal basis</u> of $L^2(\mathbb{T}^d)$.

Need to know that

$$\mathcal{A} = \left\{ \sum_{\underline{n}\in F} c_{\underline{n}} e_{\underline{n}} : c_{\underline{n}} \in \mathbb{C},\ F \subseteq \mathbb{Z}^d, |F| < \infty \right\}$$

is dense in $L^2(\mathbb{T}^d)$.

Elements of $\mathcal{A}$ are called trig. polynomi[als]

$\mathcal{A}$ is an algebra:
- containing $[\mathbb{1} = e_{\underline{0}}]$
- closed under conj. $[\overline{e_{\underline{n}}} = e_{-\underline{n}}]$
- seperating points

$\xrightarrow{\text{Stone-Weierstrass}}$ $\mathcal{A}$ is dense in $C(\mathbb{T}^d)$

Hence $\mathcal{A}$ is also dense in $L^2(\mathbb{T}^d)$.

It follows that there is an isomorphism

$$f \in L^2(\mathbb{T}^d) \longrightarrow (c_{\underline{n}})_{\underline{n} \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$$

$$c_{\underline{n}} = \langle f, e_{\underline{n}} \rangle$$

with the inverse

$$f = \underbrace{\sum_{\underline{n}} c_{\underline{n}} e_{\underline{n}}}_{\text{converging in } L^2(\mathbb{T}^d)}$$

$c_{\underline{n}}$ are called the <u>Fourier coefficients</u>

the sum $\sum_{\underline{n}} c_{\underline{n}} e_{\underline{n}}$ is called the

Fourier series of f.

<u>Note</u> We do <u>not</u> claim here that

$$f(\underline{x}) = \sum_{\underline{n}} c_{\underline{n}} e_{\underline{n}}(\underline{x})$$

for a fixed $\underline{x} \in \mathbb{T}^d$. This does not hold in general, not even when $f \in C(\mathbb{T}$

__Proposition__ If $f \in C^k(\mathbb{T}^d)$ for some $k > \frac{d}{2}$, then

$$f(x) = \sum_{\underline{n}} c_{\underline{n}} \, e_{n}(x)$$

is absolutely uniformly converging for $x \in \mathbb{T}^d$.

__Pf.__ $f \in C^1(\mathbb{T}^d)$ implies that the Fourier coefficients of f and of $\frac{\partial f}{\partial x_j}$ are related by the formula

$$\left\langle \frac{\partial f}{\partial x_j}, e_{\underline{n}} \right\rangle \underset{\uparrow}{=} -\left\langle f, \frac{\partial}{\partial x_j} e_{\underline{n}} \right\rangle = -\langle f, 2\pi i n_j e_{\cdot}$$

integration by parts [using periodicity] in the direction of x_j

$$= 2\pi i n_j \langle f, \cdot$$

where $\underline{n} = (n_1, \dots, n_d)$.

Assuming $f \in C^k(\mathbb{T}^d)$ this shows for

$$c_{\underline{n}} = \langle f, e_{\underline{n}} \rangle$$

that

$$\left(n_j^k c_{\underline{n}} \right)_{\underline{n}} \in \ell^2(\mathbb{Z}^d).$$

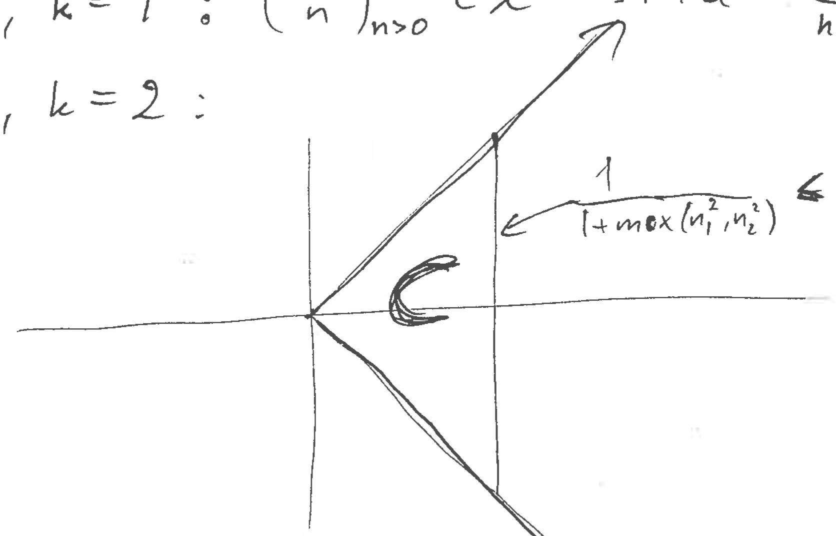
and so also $\left((1 + \max_j |n_j|^k) c_{\underline{n}} \right)_{\underline{n}} \in \ell^2(\mathbb{Z}^d)$

<u>Claim</u> If $k > \frac{d}{2}$ then

$$\frac{1}{1 + \max_j |n_j|^k} \in \ell^2(\mathbb{Z}^d)$$

<u>Pf</u>: $d = 1$, $k = 1$: $\left(\frac{1}{n}\right)_{n>0} \in \ell^2$ since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ $\square_{d=}$

$d = 2$, $k = 2$:



$$\sum_{\substack{n \in C \\ n \neq 0}} \left(\frac{1}{1 + \max_j |n_j|^2}\right)^2 \leq \sum_{\substack{n \in C \\ n \neq 0}} \frac{1}{n_1^4} = \sum_{n_1 = 1}^{\infty} \sum_{n_2 = -n_1}^{n_1} \frac{1}{n_1^4}$$

$$= \sum_{n_1 = 1}^{\infty} \frac{2n_1 + 1}{n_1^4} < \infty$$

4 cones as the one above cover $\mathbb{Z}^2 \setminus \{0\}$

$\square$ $d = 2$

Similiar splittings of $\mathbb{Z}^d$ for a general d proves the claim — check this

$\square_{d,}$ claim

The prop. follows:

$$\sum |c_{\underline{n}}| \leq \left\langle \frac{1}{1+\max_j |n_j|^k}, (1+\max_j |n_j|^k)|c_{\underline{n}}| \right\rangle_{\ell^2(\mathbb{Z}^d)} < \infty$$

by Cauchy-Schwarz

This implies uniform convergence: If $\varepsilon > 0$ then $\exists N$ s.t. $\sum_{\substack{|\underline{n}|>N \\ \underline{n} \in \mathbb{Z}^d}} |c_{\underline{n}}| < \varepsilon$. Hence if

$F_1, F_2 \subseteq \mathbb{Z}^d$ are finite sets with

$$\{\underline{n} \in \mathbb{Z}^d \mid |\underline{n}| \leq N\} \subseteq F_1 \cap F_2$$

then $\left\| \sum_{\underline{n} \in F_1} c_{\underline{n}} e_{\underline{n}} - \sum_{\underline{n} \in F_2} c_{\underline{n}} e_{\underline{n}} \right\|_\infty < \varepsilon$

which implies that $\sum_{\underline{n} \in \mathbb{Z}^d} c_{\underline{n}} e_{\underline{n}}$ converges in $C(\mathbb{T}^d)$ uniformly to some g.

The Fourier coefficients of g equal the ones of f $\Rightarrow$ $f = g$ is the uniform limit.

$\square$

Recall for $f, g \in L^1(\mathbb{T}^d)$ we can define

$$f * g\,(\underline{x}) = \int_{\mathbb{T}^d} f(\underline{t})\, g(\underline{x}-\underline{t})\, dt.$$

<u>Lemma</u> The Fourier coefficients $c_n(f*g)$ of $f*g$ are given by

$$c_n(f*g) = c_n(f)\, c_n(g)$$

by the product of the Fourier-coefficients of f, g.

<u>Pf</u> $c_n(f*g) = \iint f(\underline{t})\, \underbrace{g(\underline{x}-\underline{t})}_{g(\underline{y})}\, dt\; \underbrace{\overline{e_{\underline{n}}(\underline{x})}}_{\overline{e_{\underline{n}}(t)}\,\overline{e_n(y)}}\, d\underline{x} = c_n(f)\, c_n(g)$ $\square$

<u>Lemma</u> If $f, g \in C(\mathbb{T}^d)$ then the Fourier-coeff of $f \cdot g \in C(\mathbb{T}^d)$ are given by

$$c_{\underline{n}}(fg) = \sum_{\underline{m} \in \mathbb{Z}^d} c_{\underline{m}}(f)\, c_{\underline{n}-\underline{m}}(g)$$

<u>Pf</u> $c_0(fg) = \int f(x) g(\underline{x})\, dx = \langle f, \bar{g} \rangle \underset{\substack{\uparrow \\ \text{isomorphism} \\ \text{between} \\ L^2\ \&\ \ell^2}}{=} \langle (c_n(f))_n, (c_n(\bar{g}))_n \rangle$

What is $c_n(\bar{g}) = \int \bar{g}\, \bar{e}_n\, d\underline{x} = \overline{\int g\, e_n\, dx} = \overline{\int g\, \bar{e}_{-n}\, dx} = \overline{c_{-n}(g)}$

Hence $c_0(fg) = \sum_m c_m(f)\, c_{-m}(g) \Rightarrow$ lemma for $\underline{n} = 0$.

$c_{\underline{n}}(fg) = c_0(fg\, e_{-n}) = \sum_m c_m(f)\, \underbrace{c_{-m}(g\, e_{-n})}_{c_{n-m}(g)}$ $\square$

Definition

A sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers is called positive definite if for any finite sequence c_n of complex numbers ($c_{\underline{n}}=0$ for large $\underline{n}\in\mathbb{Z}^d$) we have

$$\sum_{n,m} c_n \overline{c_m}\, a_{n-m} \geq 0$$

(meaning $\in\mathbb{R}$ & positive)

Model-Example & Definition

Let μ be a finite measure on $\mathbb{T}$. Then the Fourier-coefficients of μ are defined by

$$a_n(\mu) = \int e_n(x)\, d\mu(\underline{x})$$

Claim The sequence $a_n(\mu)$ is positive definite.

Pf.
$$\sum_{n,m} c_n \overline{c_m}\, a_{n-m}(\mu) =$$

$$\int \sum_{n,m} c_n \overline{c_m}\, \underset{\overset{\|}{e_n \overline{e_m}}}{e_{n-m}}\, d\mu$$

$$= \int \underbrace{\left(\sum_n c_n e_n\right) \overline{\left(\sum_m c_m e_m\right)}}_{\geq 0}\, d\mu \geq 0$$

$\square$

Theorem (Bochner)

Let $(a_n)_n$ be a positive definite sequence. Then there exists a finite measure μ on $\mathbb{T}$ for which

$$a_n = a_n(\mu) = \int e_n\, d\mu.$$

This is a very useful theorem. In particular it gives rather quickly:

Corollary (Spectral theory of unitary operato

Let H be a Hilbert space. Let

$$U : H \to H$$

be a unitary map (i.e. $U^* = U^{-1}$).

Then $H = \bigoplus_n H_n$, each H_n is U-invariant,

$H_n = \overline{\langle U^n v_n : n \in \mathbb{Z} \rangle}$ for some $v_n \in H_n$, and

$H_n \cong L^2(\mathbb{T}, \mu_n)$ for a finite measure μ_n

such that

$$\boxed{U|_{H_n} \circlearrowleft H_n} \;\hat{=}\; \boxed{M_{e_1} \circlearrowleft L^2(\mathbb{T}, \mu_n)}$$

What is the connection?

Equally important example

If $U : H \circlearrowleft$, $v \in H$, then

$$a_n = \langle U^n v, v \rangle$$

is positive definite.

Pf. of Example

Suppose $c_n \in \mathbb{C}$ finite,

$$\sum_{m,n} c_n \overline{c_m}\ a_{n-m} = \sum_{m,n} c_n \overline{c_m} \langle U^{n-m} v, v\rangle$$

$$= \sum_{m,n} \langle c_n U^n v, c_m U^m v\rangle$$

$$= \left\| \sum c_n U^n v \right\|^2 \geq 0$$

$\square$

Pf of Cor assuming Thm. Let $v \in H \setminus \{0\}$

By the above

$$a_n = \langle U^n v, v\rangle$$

is pos. def. By the Thm

$$a_n = \int e_n \, d\mu$$

for a finite measure μ.

We need to define an isomorphism from

$$\overline{\langle U^n v : n \in \mathbb{Z}\rangle} \xrightarrow{\Phi} L^2(\mathbb{T}, \mu).$$

Definition

$$\phi\Big(\underbrace{\sum_n}_{\text{finite}} c_n U^n v\Big) = \sum_n c_n e_n \in L^2(\mathbb{T})$$

Problems: • Why well-defined?
• Why can this be extended to $\overline{\langle \dots \rangle}$
• Why does it give an isometry?

Note

$$\Big\|\sum_n c_n U^n v\Big\|^2 = \sum_{m,n} \langle c_n U^n v, c_m U^m v\rangle$$

$$= \sum_{m,n} c_n \overline{c_m} \underbrace{\langle U^{n-m} v, v\rangle}_{= a_{n-m}}$$

$$= \sum_{m,n} c_n \overline{c_m} \int e_{n-m} \, d\mu$$

$$= \int \Big|\sum_n c_n e_n\Big|^2 d\mu$$

which implies that ϕ as above preserves the norm. However, this also shows that it is well-defined:

If $\sum_n c_n U^n v = \sum_n c_n' U^n v$ (which may happen in H, then $\sum (c_n - c_n') U^n v = 0$ and so has norm $= 0$. By the above $\sum (c_n - c_n') e_n \in L^2(\mu)$ also has norm 0, so that $\sum c_n e_n = \sum c_n e_n'$ (mod μ) as required.

$\Rightarrow \phi$ is well-defined, linear, isometric. By the latter ϕ also extends uniquely to the closure of all finite sums.

<u>Claim</u> This extended ϕ is also onto.

$\operatorname{Im}\phi \subseteq L^2(\mu)$ is closed (as it is isomorphic to $\overline{\langle\,\rangle} \subseteq H$ it is complete), and contains all trig. polynomials, i.e. $\mathcal{A}$. Hence $C(\mathbb{T}) \subseteq \operatorname{Im}\phi$ and finally $L^2(\mu) = \operatorname{Im}\phi$.

The subspaces H_n are defined <u>recursively</u>:

let $v_1 \in H \setminus \{0\}$ define $H_1 = \langle \phi^n v : n \in \mathbb{Z} \rangle$

let $v_2 \in H_1^\perp \setminus \{0\}$ (if it exists), - - - - - -

$\square$

We are now well-motivated to prove the theorem; for this we need some preparations:

<u>lemma</u> If a_n is a pos-definitive sequence then

- $a_0 \geq 0$
- $a_{-k} = \overline{a_k}$ for $k \in \mathbb{Z}$
- $|a_k| \leq a_0$ for $k \in \mathbb{Z}$

<u>Pf.</u>

- let $c_n = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases} \Rightarrow \sum_{n,m} c_n \overline{c_m} a_{n-m} = a_0 \geq 0$

- let $c_n = \begin{cases} 1 & n=0 \\ x & n=k \\ 0 & n \neq 0, \text{ or } k \end{cases}$

$$\Rightarrow \sum_{n,m} c_n \overline{c_m} a_{n-m} = \underset{\substack{(n=m=0) \\ \in \mathbb{R}}}{a_0} + \underbrace{\underset{(n=k,m=0)}{x a_k} + \underset{(n=0,m=-k)}{\overline{x} a_{-k}}} + \underset{\substack{(n=m=k \\ \in \mathbb{R}}}{|x|^2 a_0} \overset{\in}{\geq}$$

$\Rightarrow$, by setting $x=1$, $a_k + a_{-k} \in \mathbb{R}$ & so $\mathrm{Im}(a_{-k}) = -\mathrm{Im}($

by setting $x=i$, $i a_k - i a_{-k} \in \mathbb{R}$ & so $\mathrm{Re}(a_k) = \mathrm{Re}(a$

$$\Rightarrow a_{-k} = \overline{a_k}$$

• now write $a_k = |a_k| e^{i\theta}$

and set $x = r e^{-i\theta}$ for $r \in \mathbb{R}$

then the above gives

$$a_0 + 2r|a_k| + r^2 a_0 \geq 0$$

for all $r \in \mathbb{R}$

if $a_0 = 0 \Rightarrow 2r|a_k| \geq 0$ shows $a_k = 0$

if $a_0 \neq 0$ then the discriminant of the above q.f. in r must be non-positive

$$4|a_k|^2 - 4\, a_0 \cdot a_0 \leq 0$$

$$\Rightarrow |a_k| \leq a_0$$

$\square$

<u>Pf of Thm.</u>

Given a pos. def. seq. a_n we need to define a measure μ on $\mathbb{T}$ with $a_n = \int e_n \, d\mu$.

Naively: To define μ we need to define $\Lambda(f) = \int f \, d\mu$. Clearly

$$\lambda(e_n) = a_n$$

so we try to define

$$\Lambda(f) = \sum_n c_n a_n$$

for $f = \sum c_n e_n$.

The problem with this is:

$$f \in C(\mathbb{T}) \Rightarrow f \in L^2(\mathbb{T}) \Rightarrow (c_n) \in \ell^2(\mathbb{Z})$$

but not $(c_n) \in \ell^1(\mathbb{Z})$.

We start with a definition of

$$\Lambda(f) \text{ for } f \in C^1(\mathbb{T}).$$

By our earlier discussion (Prop reg. abs. unif. convergence) $f \in C^1(\mathbb{T})$ implies $(c_n) \in \ell^1(\mathbb{Z})$ and

so

$$f \mapsto \Lambda(f) = \sum_n c_n a_n$$

is well-defined and linear since $a_n \in \ell^\infty(\mathbb{Z})$ by the above lemma.

need to show: • Λ extends to $C(\mathbb{T})$

• $\Lambda(f) \geq 0$ if $f \geq 0$

<u>Weaker Claim:</u> $f \in C^1(\mathbb{T})$, $f > 0$

$\Rightarrow \Lambda(f) \geq 0$

<u>Pf of Claim</u>

$f \in C^1(\mathbb{T})$, $f > 0 \Rightarrow \sqrt{f} \in C^1(\mathbb{T})$

by chain rule

let (c_n) be the Fourier-coeff. of $\sqrt{f}$.

Then the Fourier-coeff of f

are given by

$$c_n(f) = \sum_m c_m c_{n-m},$$

by an earlier lemma.

Therefore,

$$\Lambda(f) = \sum_n c_n(f) a_n = \sum_n \left(\sum_m c_m c_{n-m} \right) a_n.$$

Note $(c_n) \in \ell^1(\mathbb{Z}) \Rightarrow (c_m c_{n-m}) \in \ell^1(\mathbb{Z} \times \mathbb{Z})$

$(a_n) \in \ell^\infty(\mathbb{Z} \times \mathbb{Z})$

$$= \sum_{m,n} c_m c_{n-m} a_n$$

Note $\hat{f} > 0 \Rightarrow c_n = \overline{c_n}$

So that

$$\Lambda(f) = \sum_{m,n} c_m \overline{c_{\underbrace{m-n}_{k}}}\, a_n$$

$$= \underbrace{\sum_{m,k} c_m \overline{c_k}\, a_{m-k}}_{\text{converging double-sum}}$$

by def. of pos. def. we have

$$\sum_{|m|,|k|<N} c_m \overline{c_k}\, a_{m-k} \geq 0$$

& so also $\Lambda(f) \geq 0$.

$\square$ of weaker Claim.

<u>Claim</u> $f \in C^1(\mathbb{T}),\ f \geq 0 \Rightarrow \Lambda(f) \in [0, \|f\|_\infty a_0]$

$f \geq 0 \Rightarrow f + \varepsilon > 0 \Rightarrow \Lambda(f+\varepsilon) = \Lambda(f) + \varepsilon a_0 \geq 0 \quad \forall \varepsilon > 0$

$f \leq \|f\|_\infty \Rightarrow \|f\|_\infty - f \geq 0 \Rightarrow \Lambda(f) \leq \Lambda(\|f\|_\infty) = a_0 \|f\|_\infty$

$\square$ of Claim

Claim $f \in C^1(\mathbb{T})$, real valued

$$\Rightarrow |\Lambda(f)| \leq a_0 \|f\|_\infty$$

Pf. $\pm f + \|f\|_\infty \geq 0 \Rightarrow \pm \Lambda(f) + a_0 \|f\|_\infty \geq 0$ $\square$

This shows Λ is cont. w.r.t. $\| \|_\infty$, and since $C^1(\mathbb{T}) \subseteq C(\mathbb{T})$ is dense w.r.t. $\| \|_\infty$, Λ extends to $C(\mathbb{T})$. Moreover, if $f \in C(\mathbb{T})$, $f \geq 0$ then there exists $f_n \in C^1(\mathbb{T})$ with $f_n \to f$ uniformly so that $f_n \geq -\|f_n - f\|_\infty$ implies

$$\begin{array}{ccc} \Lambda(f_n) & \geq -a_0 & \|f_n - f\|_\infty \\ \downarrow & & \downarrow \\ \Lambda(f) & \geq & 0 \end{array}$$

$\Rightarrow \Lambda(f) = \int f \, d\mu$ for a finite measure

$$\Lambda(e_n) = a_n = \int e_n \, d\mu$$ as required.

$\square$

Comments

- Bochner's theorem and its proof generalize to $\mathbb{T}^d$ without many changes

- Fourier series can be generalized to any compact abelian gp.
  E.g. $\mathbb{Z}_2$ = group of p-adic integers
  or $\mathbb{F}_p[[t]] \cong \mathbb{F}_p^{\mathbb{N}}$
  The characters are the continuous homomorphisms from the gp to
  $$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

- Bochner's thm generalizes to compact abelian gps — for these one needs to rethink the pf.