Fa 2

ex: $U: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$v = (v_1, v_2, \ldots) \to U(v) = (0, v_1, v_2, \ldots)$ has no eigenvectors

$U(v) = \lambda v \Rightarrow (U(v))_1 = 0 = \lambda v_1$ $\to \lambda = 0 \Rightarrow v = 0$ $*$
$\to v_1 = 0 - - v = 0$. $*$

1. Compact operators

Def: A bdd operator $A: V_1 \to V_2$ between Banach spaces is compact if

$$\overline{A(\{v \in V_1 : \|v_1\| \le 1\})}^{V_2} \text{ is compact.}$$

Thm: A Banach space V satisfies that $\{v \in V : \|v\| \le 1\}$ is compact iff V is finite dimensional.

Thm: (Arzela-Ascoli)
If X cpct m.sp, & $K \subset C(X)$ which is closed bdd, & equicontinuous then every sequence $f_n \in K$ has a conv. subsequence

Def. A subset $K \subseteq C(X)$ is equicontinuous if
$\forall \varepsilon > 0\ \exists \delta > 0$ st $\forall f$, $\forall x, y \in X$ $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

ex: If $A: V_1 \to V_2$ has finite-dimensional image $A(V_2)$ & A is bdd $\to$ A is cpct.

Prop: If V_1, V_2 are B-spaces. If $A_n: V_1 \to V_2$ are cpct operators for $n = 1, 2, \ldots$ & if $A_n \to A$ in the norm on $B(V_1, V_2)$ then A is cpct.

Pf: Let $\varepsilon > 0$. let n be such that $\|A - A_n\| < \varepsilon$ let $v_1, \ldots, v_m \in V_1$ with $\|v_i\| \le 1$ & st. $A_n(v_1), \ldots, A_n(v_m) \in A_n(\{v \in V_1 : \|v\| \le 1\})$ is ε-dense.

claim: $A(v_1), \ldots, A(v_m)$ is 3ε-dense in $A(\{v \in V, \mid \|v\| \le 1\})$

Ex: Convolution operators. $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [0,1)$

$f \in L^2(\mathbb{T}, m)$
↑ lebesgue

$C_f : g \in L^2(\mathbb{T}) \to f * g$ $(f*g)(x) = \int_{\mathbb{T}} f(t)\, g(x-t)\, dt$.

$C_f : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is compact.

$\searrow C(\mathbb{T}) \nearrow$ bdd inc.

is cpct. $\|g\|_2 \le \|g\|_\infty$

Pf:

$\int |f(t) g(x-t)|\, dt \le \|f\|_2 \|g\|_2 < \infty \Rightarrow \|f*g\|_\infty \le \|f\|_2 \cdot \|g\|_2$.

$f*g\,(x) = g*f\,(x)$

$\forall \varepsilon > 0 \ \exists f_1 \in C(\mathbb{T})$ s.t. $\|f - f_1\|_2 \le \varepsilon$

$\|f*g - f_1 * g\|_\infty = \|(f - f_1) * g\|_\infty \le \|f - f_1\|_2 \cdot \|g\|_2 < \varepsilon \|g\|_2$

claim: $f_1 * g$ is cts. $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon, f_1)$ st $d(x,y) < \delta \to |f_1(x) - f_1(y)| < \varepsilon$.

Pf: $|f_1 * g(x) - f_1 * g(y)| = \left| \int g(t) [\underbrace{f_1(x-t) - f_1(y-t)}_{<\varepsilon}]\, dt \right| \le \varepsilon \|g\|_2$.

$\Rightarrow f_1 * g$ is cts.

$C_{f_1} : L^2(\mathbb{T}) \to C(\mathbb{T})$ is cpct
$g \mapsto f_1 * g$

If $f_n \in C(\mathbb{T})$, $f_n \to f$ in L^2 $\to C_{f_n} \to C_f$ ▫

ex: Integral operators

$(X, \mu), (Y, \nu)$ prob spaces. $k(x,y) \in L^2(X \times Y, \mu \times \nu)$ kernel.

$K(f)(y) = \int_X f(x) k(x,y)\, d\mu(x)$ for $f \in L^2(X, \mu)$ ν-a.e. y

$Kf \in L^2(Y, \nu)$, $K : L^2(X, \mu) \to L^2(Y, \nu)$ is compact

Pf:

$$x \mapsto \underset{k(\cdot, y)}{k(x,y)}$$

$$\|k(\cdot, y)\|_2 = \left(\int_X |k(x,y)|^2 d\mu(x) \right)^{1/2}$$

$$\int_Y \|k(\cdot, y)\|_2^2 \, d\nu(y) = \|k(\cdot, \cdot)\|_2^2 < \infty$$

$$\|Kf\|_2^2 \le \int_Y \left| \int_X f(x) \, k(x,y) \, d\mu(x) \right|^2 d\nu(y)$$

$$\le \int_Y \|f\|_2^2 \cdot \|k(\cdot, y)\|_2^2 \, d\nu(y) = \|f\|_2^2 \cdot \|k\|_2^2 < \infty.$$

$\|K\| \le \|k\|_2$

claim: K is cpct

1) Suppose $k(x,y) = 1_{A \times B}(x,y) = 1_A(x) \, 1_B(y)$

$K(f) \in \text{span}(1_B)$ is cpct - f.dim. range.

2) Also cpct if k a simple fn built from rectangles

3) If k is arbitrary then $\exists$ k_n as in (2) w $\|k - k_n\|_2 \to 0$

thus $\|K - K_n\| \to 0$

2. Spectral theory of cpct self-adjoint operators

Thm: Let H be a Hilbert space (∞-dim, separable), let $A: H \to H$ be a cpct self-adj. operator. Then $\exists$ a sequence (λ_n) of real eigenvalues & an orthonormal basis v_n of H with $A v_n = \lambda_n v_n$. Moreover, $\lambda_n \to 0$.

Lemma: If $A: H \to H$ is a bdd operator & $V \subseteq H$ is a subspace with $A(V) \subseteq V$ then $A^*(V^\perp) \subseteq V^\perp$

Pf: $v \in V$, $v^\perp \in V^\perp$ $\langle A^* v^\perp, v \rangle = \langle v^\perp, Av \rangle = 0$ $\square$

~~Remark (Alternative approach)~~

Pf:

Lemma: A bd, self adj $\Rightarrow \sup_{\|x\|=1} |\langle Ax, x\rangle| = \|A\|$.

Pf: $x \in H$, $\|x\| \le 1$ $\quad \|Ax\| \le \|A\|$.

$|\langle Ax, x\rangle| \le \|Ax\|\|x\| \le \|A\|$.

Let $\lambda > 0$. $\langle A(\lambda x + \frac{1}{\lambda} Ax), \lambda x + \frac{1}{\lambda} Ax\rangle - \langle A(\lambda x - \frac{1}{\lambda} Ax), \lambda x - \frac{1}{\lambda} Ax\rangle \le$

$4\|Ax\|^2 =$

$s = \sup_{\|z\| \le 1} |\langle Az, z\rangle|$ $\quad \le s[\|\lambda x + \frac{1}{\lambda} Ax\|^2 + \|\lambda x - \frac{1}{\lambda} Ax\|^2]$

$\|$

$2(\|u\|^2 + \|v\|^2) = \|u+v\|^2 + \|u-v\|^2 \Rightarrow$ $\quad = 2s[\|\lambda x\|^2 + \|\frac{1}{\lambda} Ax\|^2]$

Let $\lambda^2 = \frac{\|Ax\|}{\|x\|} \Rightarrow 4\|Ax\|^2 \le 2s\, \|Ax\|\|x\| + \|Ax\|\|x\|$

$\Rightarrow \|Ax\| \le s\|x\|$, $\Rightarrow \|A\| \le s$.

Lemma: A cpt self-adj. on Hilbert sp H. Then $\exists$ an eigenvector $v \in H$ with eigenv. $\|A\|$ or $-\|A\|$.

Pf: $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle| \Rightarrow$ may choose a seq. $x_n \in H$, $\|x_n\| = 1$

$\langle Ax_n, x_n\rangle \to \alpha$ where $\alpha \in \mathbb{R}$, $|\alpha| = \|A\|$

$\|Ax_n - \alpha x_n\|^2 = \langle Ax_n - \alpha x_n, Ax_n - \alpha x_n\rangle$

$= \|Ax\|^2 - 2\alpha \underbrace{\langle Ax_n, x_n\rangle}_{\to \alpha} + \alpha^2 \|x_n\|^2 \le \underbrace{\|A\|^2 + \alpha^2}_{2\alpha^2} - 2\alpha\langle Ax_n, x_n\rangle$

$\to 0$

$\therefore \|Ax_n - \alpha x_n\|^2 \to 0$

$\Rightarrow \exists\, n_k$ with $Ax_{n_k} \to \alpha x$

$-Ax_{n_k} + \alpha x_n \to 0$

$\alpha x_{n_k} \to \alpha x \Rightarrow x_{n_k} \to x \Rightarrow Ax_{n_k} \to Ax = \alpha x$.

$\|x\| = 1$

□

Pf of thm: $A: H \to H$ cpct self-adj

$\Rightarrow \exists x_1 \in H$, $\|x_1\| = 1$ & $Ax_1 = \alpha_1 x$, $\alpha_1 = \pm\|A\|$

Let $V_1 = \text{span}\{x_1\} \Rightarrow A(V_1) \subseteq V_1 \Rightarrow A^*(V_1^\perp) \subseteq V_1^\perp$

$$\overset{\shortparallel}{A(V_1^\perp)}.$$

$A|_{V_1^\perp}: V_1^\perp \to V^\perp$ is selfadj, cpct $\Rightarrow \exists x_2 \in V_1^\perp$, $\|x_2\| = 1$, $A(x_2) = \alpha_2 x$

$V_2 = \text{span}\{x_1, x_2\}$, $A(V_2) \subset V_2 \Rightarrow A(V_2^\perp) \subseteq V_2^\perp$ ...

claim: $\alpha_n \to 0$

know $|\alpha_n| \downarrow$.

In this case, $Ax_n = \alpha_n x_n$ has norm $|\alpha_n| \geq \alpha > 0$

$\Rightarrow \|Ax_n\| \not\to 0$. but Ax_n conv. weakly to zero

If ker A is nontrivial this may not have produced a basis. In this case, let $y_1, \ldots \in (\ker A) \cap \{x_1, x_2, \ldots\}^\perp$ be an orth. basis of this subspace...

3. The inverse of the Laplace operator.

$\mathbb{R}^n$, $\mathbb{T}^n$

$$\Delta f = \left(\frac{\partial^2}{(\partial x_1)^2} + \cdots + \frac{\partial^2}{(\partial x_n)^2}\right) f$$

$f \in L^2$? Δ is an unbdd operator on L^2.

claim: $\Delta^{-1}: L_0^2(\mathbb{T}^n) \to L_0^2(\mathbb{T}^n)$ cpct selfadj op.

$$L_0^2 = \{f \in L^2 \mid \textstyle\int f = 0\}$$

$e_{\underline{m}}(x) = \exp(2\pi i(m_1 x_1 + \ldots + m_n x_n))$ is an eigenfn of Δ

$$\Delta(e_{\underline{m}}) = \underbrace{-4\pi^2 \|m\|^2}_{\text{eigenv.}} e_m$$

$$\Delta^{-1}\Big(\sum_{\underline{m}\neq 0} c_m \underline{e_m}\Big) = \sum_{\underline{m}\neq 0} \underbrace{\frac{1}{-(2\pi)^2\|m\|^2}}_{\to 0} c_m e_m .$$

Lemma: $\{f\in L^2(\mathbb{T}^n) : \|\frac{\partial}{\partial x_i} f\|_2 \le 1,\ \|f\|_2 \le 1\}$ is compact.

Lemma: In ℓ^2 the set $\underbrace{\{(x_n)\in \ell^2 \mid \sum c_n |x_n|^2 \le 1\}}_{K}$ is cpct if $c_n \ge 1$ & $c_n \to \infty$

Pf: let $\varepsilon>0$ & let N be s.t. $c_n \ge \frac{1}{\varepsilon}$ for $n\ge N$.

$\Rightarrow (x_n)\in K \Rightarrow \|(x_1,x_2,\dots,x_N,x_{N+1},\dots) - (x_1,\dots,x_N,0,0,\dots)\|$

$= \sum_{n\ge N} |x_n|^2 \le \varepsilon \sum_{n>N} c_n |x_n|^2 \le \varepsilon$

In $\mathbb{C}^N$ the expression $\sum c_n |x_n|^2 \le 1$ defines a cpct subset.

hence $\exists$ a finite list of ε-dense vectors $y^{(1)},\dots,y^{(m)}$

$\Rightarrow$ any $(x_n)\in K$ is 2ε-close to some $(y_n^{(i)})$ $\square$

Lemma: $f,g\in C^\infty(\mathbb{T}^d)$

$\langle f, \frac{\partial}{\partial x_i} g\rangle_{L^2} = -\langle \frac{\partial}{\partial x_i} f, g\rangle_{L^2}$ $\qquad \frac{\partial}{\partial x_i}$ is anti-selfadjoint.

Pf: int. by parts.

Integrate now over $x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_d$ $\square$

Def: $f\in L^2(\mathbb{T}^d)$ has a partial derivative $f_j \in L^2(\mathbb{T}^d)$ (in the sense of distributions) along x_j if

$$\langle f, \frac{\partial}{\partial x_j}\varphi\rangle_{L^2} = -\langle f_j, \varphi\rangle_{L^2} \quad \forall \varphi\in C^\infty(\mathbb{T}^d)$$

Rk: If f_j exists it is unique.

Lemma: If $f_j\in L^2$ is the part. der of $f\in L^2$ &

$f = \sum_{\underline{n}\in\mathbb{Z}^d} c_n e_{\underline{n}}$ then $f_j = \sum_{\underline{n}\in\mathbb{Z}^d} c_n e_{\underline{n}} (2\pi i n_j)$

Pf: $(f_j, e_n) = -(f, 2\pi i n_j e_n) = 2\pi i n_j \underbrace{(f, e_n)}_{c_n}$ $\square$

Lemma: (compactness) The set $K = \{ f \in L^2(\mathbb{T}^d) :$ f has the part. der. $f_j \in L^2(\mathbb{T}^d)$ along x_j & $\|f\|_2 \le 1$, $\|f_j\|_2 \le 1$ $\forall j = 1, \dots, d \}$

is compact.

Pf: $K \subseteq \{ f = \sum c_n e_n \in L^2 \mid \sum |c_n|^2 (1 + (2\pi)^2 \|n\|^2) \le d+1 \}$

is cpct by first lemma today. $\square$

open, conn.

Extend this to bdd domains in $\mathbb{R}^d$

$C_c^\infty(D) = \{ f : D \to \mathbb{C} \mid f$ is smooth & has cpct support $\}$.

Def: $f \in L^2(D)$ has a partial derivative f_j along x_j if

$$\langle f, \frac{\partial}{\partial x_j} \varphi \rangle_{L^2} = - \langle f_j, \varphi \rangle_{L^2} \quad \forall \varphi \in C_c^\infty(D)$$

Lemma: If $f \in C^\infty(D) \cap L^2$ with bdd derivative $\Rightarrow$ the derivative satisfies the def.

- The standard inner product

$$\langle f, g \rangle_0 = \int_D f \bar{g} \, dx \quad \text{for } f, g \in C_c^\infty(D)$$

The completion is $L^2(D)$ $\qquad \varphi$, $f \approx f * \varphi$

- Lax, pg 65

The inner product $\langle \, , \rangle_1$

$$\langle f, g \rangle_1 = \int_D \sum_j \left(\frac{\partial f}{\partial x_j}\right) \overline{\left(\frac{\partial g}{\partial x_j}\right)} \, dx$$

The associated (semi-) norm is

$$\|f\|_1^2 = \sum_j \left\| \frac{\partial f}{\partial x_j} \right\|_{L^2}^2$$

Prop: $D \subseteq \mathbb{R}^d$ bdd domain

We have

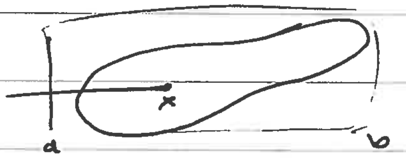
$$\|f\|_0 \leq \text{width}(D)\, \|f\|_1$$

Hence. $\exists$ a cts map from the completion H_1 of $C_c^\infty(D)$ w.r.t $\|\cdot\|_1$

to $H_0 = L^2$

$$\iota : H_1(D) \to H_0(D)$$

Pf:



$$f(x_1, \dots, x_d) = \int_a^{x_1} \frac{\partial}{\partial x_1} f(t, x_2, \dots, x_d)\, dt$$

$$|f(\underline{x})|^2 \leq \int_a^{x_1} \left|\frac{\partial}{\partial x_1} f\right|^2 dt \cdot \underbrace{\text{width}}_{b-a}$$

$$\int_a^{\cancel{x}} |f|^2 dx_1 \leq \int_a^x \int_a^{x_1} \left|\frac{\partial}{\partial x_1} f\right|^2 dt\, dx_1 \cdot (b-a)$$

$$\int_a^b |f|^2 dx_1 = \int_a^b \int_a^b \left|\frac{\partial}{\partial x_1} f\right|^2 \chi_{\{t < x_1\}}\, dx_1\, dt \quad (b-a)$$

$$\int_a^b \chi_{\{t<x_1\}}\, dx_1 \leq b-a$$

$$\leq \cancel{\int_a} (\text{width})^2 \int_a^b \left|\frac{\partial}{\partial x_1} f\right|^2 dt$$

$$\Rightarrow \|f\|_2^2 \leq (\text{width})^2 \left\|\frac{\partial}{\partial x_1} f\right\|_2^2 \qquad \square$$

ι is injective:

claim $F \in H_1$ & $(\iota(F))$ has weak partial derivatives

in $L^2(D)$

If $f \in C_c^\infty(D) \hookrightarrow \left(\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \dots, \frac{\partial f}{\partial x_d}\right) \in (L^2(D))^d$

$\|f\|_1 = \|$ " $\|_{L^2(D)^d}$

So, H_1 is the closure of C_c^∞ inside $(L^2(D))^d$.

~~If $\varphi \in C_c^\infty(D)$, $f_k \in C_c^\infty(D)$, $Df_k \to (F_1, \dots, F_d)$, $F_j \in L^2$~~

$$\langle f_k, \frac{\partial}{\partial x_j}\varphi\rangle = -\langle \frac{\partial}{\partial x_j} f_k, \varphi\rangle$$

$$\downarrow \qquad\qquad \downarrow$$

$$\langle F, \frac{\partial}{\partial x_j}\varphi\rangle = -\langle F_j, \varphi\rangle \qquad \forall \varphi \in C_c^\infty(D)$$

~~If $\iota(F) = 0$ then $\langle \iota(F), \frac{\partial}{\partial x_j}\varphi\rangle = 0 = -\langle F_j, \varphi\rangle \Rightarrow$~~

~~$F_j = 0 \ \forall j$~~ $\Rightarrow (F_1, \dots, F_d) = 0$.

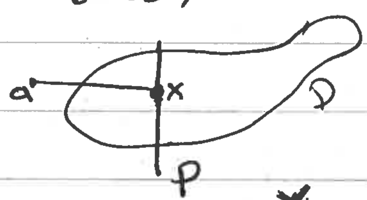
$$C_c^\infty(D) \to H_0 = L^2(D)$$

$$\searrow H_1 \subseteq L^2(D)$$

$\iota : H_1(D) \to H_0(D)$ injective

<u>Lemma</u>: Let $D \subseteq \mathbb{R}^d$ bdd open, Let $P \subset \mathbb{R}^d$ be a hyperplane. Then $\underset{H_1}{\overset{\cap}{f}} \to f|_{D\cap P} \in L^2(D\cap P)$ is a well-defined bdd map.

<u>Pf</u>: $f \in C_c^\infty(D)$



$$f(x) = \int_a^{x_1} \frac{\partial f}{\partial x_1}(t, x_2, x_3, \dots, x_d)\,dt$$

$$|f(x)|^2 \le \text{width} \int_a^{x_1} \left|\frac{\partial f}{\partial x_1}\right|^2 dt$$

$$\int_P |f(x)|^2_{dx_2 \dots dx_d} \le \text{width} \int_{\substack{\text{left side of}\\ D}} \left|\frac{\partial f}{\partial x_1}\right|^2 dx_1 \dots dx_d.$$

$$\| f|_{D\cap P} \|_{L^2} \le \sqrt{\text{width}}\ \underbrace{\left\|\frac{\partial f}{\partial x_1}\right\|_{L^2}}_{\le \|f\|_1}$$

Thm: There exists an operator

$S : H_0 \to H_1$

$S := -\iota^*$

S satisfies for every $\varphi \in C_c^\infty(D)$ that $\{\varphi, f\}_0 = \langle \Delta\varphi, S(f)\rangle_0$ "is" $\langle \Delta\varphi, \iota S f\rangle_0$

Moral: As Δ is "selfadj", this is saying that

$f = \Delta \circ S(f)$

Pf: let $f \in H_0$, $\varphi \in C_c^\infty(D)$

$\{\varphi, f\}_0 = \langle \iota(\varphi), f\rangle_0 = \langle \varphi, -S(f)\rangle_1 = -\sum_{j=1}^{d} \langle \frac{\partial\varphi}{\partial x_j}\varphi, S(f)_j \rangle_{L^2}$

$= \sum_{j=1}^{d} \langle \frac{\partial^2}{\partial x_j^2}\varphi, S(f)\rangle$ b/c $S(f)_j$ is an L^2-deriv. of $S(f)$

Cor: $\iota \circ S : H_0(D) \to H_0(D)$ is a self-adj. operator.

Pf: $(\iota\circ S)^* = S^* \circ \iota^* = -(-\iota)\circ S = \iota \circ S$. □

Thm: If D is sufficiently smooth then $\iota : H_1(D) \to H_0(D)$ is a compact operator.

In particular, $\iota\circ S$ is a compact self-adj operator.

$\Rightarrow$ gives eigenfs of Δ (but this needs more work!)

Lemma: $\bar{D} = [a,b]^d \subset \mathbb{R}^d$

For any $\eta > 0$ $\{f \in L^2(D) : f$ has partial derivatives $f_j \in L^2(D)$ along x_j, $\|f\|_{L^2}, \|f_j\|_{L^2} \le 1$ $\forall j$ &

$\text{supp}(f) \subset [a+\eta, b-\eta]^d\} =: K$ is cpct.

Pf: Suppose $a=0, b=1$

$\langle f, \frac{\partial}{\partial x_j}\varphi\rangle_{L^2} = -\langle f_j, \varphi\rangle_{L^2}$ $\forall \varphi \in C_c^\infty$

claim: when we periodically continue $f, f_1, \ldots, f_d$ to $\mathbb{T}^d$ then $f_j \in L^2(\mathbb{T}^d)$ is the part. der. of $f \in L^2(\mathbb{T}^d)$ - where of course $\varphi \in C^\infty(\mathbb{T}^d)$ is allowed pf of claim: $\exists \psi : \mathbb{T}^d \to [0,1]$

$\psi \equiv 1$ on $[\eta, 1-\eta]^d$

$\psi \equiv 0$ on $\mathbb{T}^d \setminus [\eta/2, 1-\eta/2]^d$

take $\varphi \in C^\infty(\mathbb{T}^d)$

$$\langle f, \frac{\partial}{\partial x_j} \varphi \rangle_{L^2(\mathbb{T}^d)} = \langle f, \frac{\partial}{\partial x_j} (\underbrace{\psi \varphi}_{\psi\varphi|_D \in C_c^\infty(D)}) \rangle_{L^2(D)} = - \langle f_j, \psi\varphi \rangle_{L^2(D)} = - \langle f_j, \varphi \rangle_{L^2(\mathbb{T}^d)}$$

$\Rightarrow$ done by lemma from monday $\square$

Lemma (Product Rule) $\psi \in C_c^\infty(D)$, $f \in H_1(D)$ $\Rightarrow$

$f \cdot \psi \in H_1(D)$, $(f \cdot \psi)_j = f_j \psi + f \psi_j$, $\|f \cdot \psi\|_1 \leq \|f\|_1 \|\psi\|_\infty + \|L(f)\|_2 \, d^{1/2} \cdot \max_{j=1}^{d} \|\frac{\partial}{\partial x_j} \psi\|_\infty$

Pf: When $f \in C_c^\infty(D)$,

$$(f_1, \dots, f_d)\psi + f \underbrace{\left(\frac{\partial}{\partial x_1}\psi, \dots, \frac{\partial}{\partial x_d}\psi\right)}_{\|\cdot\|_2 \leq \sqrt{\|f \frac{\partial}{\partial x_1}\psi\|_{L^2}^2 + \dots + \|f \frac{\partial}{\partial x_d}\psi\|^2}} \leq M \sqrt{\|f\|_{L^2}^2 + \dots + \|f\|_{L^2}^2}$$

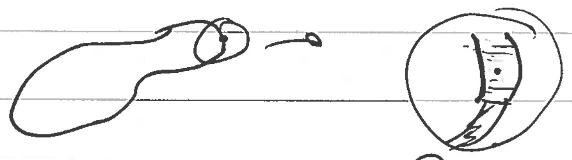
this extends to the completion.

Lemma: (L^2 bd for values close to bdry)

$D \subset \mathbb{R}^d$ open bdd, suff. smooth, $\partial D(\eta) = \{x \in D \mid \operatorname{dist}(x, \partial D) < \eta\}$

$$f \in H_1(D) \Rightarrow \|L(f)|_{\partial D(\eta)}\|_{L^2(\partial D(\eta))} \ll_D \eta \|f\|_1$$

Morally: a fn like  must have large $\|\cdot\|_1$-norm.

Pf:  Same proof with width repl. by η.

$\partial D(\eta)$

Lemma: If ∂D is sufficiently smooth, then for every $\eta > 0$ $\exists\, \psi$, $\psi \equiv 0$ on $\partial D(\eta)$, $\psi \equiv 1$ on $D \setminus \partial D(2\eta)$, $\|\psi\|_\infty \leq 1$

$$\left\| \frac{\partial \psi}{\partial x_j} \right\|_\infty \ll_D \frac{1}{\eta}.$$

$\square$

Lemma: $f \in H_1(D) \Rightarrow \quad \| \iota(f)\psi - \iota(f) \|_0 \ll \eta \|f\|_1$

$\| f\psi \|_1 \leq \|f\|_1$