29.03.2010 Mon.

We are discussing §7 Unbdd operator.:

T from H to H' def. on a dense domain $\mathcal{D}(T)$ & linear there. $\Rightarrow T^*$ from H' to H def. on $\mathcal{D}(T^*)$ & linear there.

we assume that T is a closed operator. i.e. Graph $(T) \subseteq H \times H'$ is closed. Then T^* is closed.

* $H \times H' = \text{Graph}(T) \oplus \overline{\text{Graph}(T^*)}$

Def: A closed operator $T: H \to H$ with dense domain is called self-adjoint if $T^* = T$ & $\mathcal{D}(T^*) = \mathcal{D}(T)$.

Model example: $H = L^2(X, \mu)$ ← some σ-finite measure space.

let $g: X \to \mathbb{C}$ measurable. We can define a closed operator M_g from H to H:

$$\mathcal{D}(M_g) = \{ f \in L^2(X,\mu) \mid \int_X |g|^2 |f|^2 d\mu < \infty \}$$

$$M_g(f) = gf \quad \text{for } f \in \mathcal{D}(M_g).$$

Claim: M_g is a closed operator from H to H, with dense domain

Pf: If $f_1, f_2 \in \mathcal{D}(M_g) \Rightarrow \alpha f_1,\ f_1 + f_2 \in \mathcal{D}(M_g) \Rightarrow \mathcal{D}(M_g)$ is a subspace & M_g is linear.

Dense domain: Let $f \in L^2(X,\mu)$, then

$$f_{\leq M}(x) := \begin{cases} f(x) & \text{if } |g(x)| \leq M \\ 0 & \text{if } |g(x)| > M \end{cases}$$

satisfies $\|f - f_{\leq M}\|_2 \to 0$ as $M \to \infty$:

$$\|f - f_{\leq M}\|_2^2 = \int_{\{|g|>M\}} |f|^2 d\mu = \int_X \underbrace{\chi_{\{|g|>M\}}}_{\to 0 \text{ pointwise}} |f|^2 d\mu$$

thus by dominated convergence $\|f - f_{\leq M}\|_2^2 \to 0$.

Also, $f_{\le m} \in \mathcal{D}(M_g)$. Thus $\mathcal{D}(M_g)$ is dense in L^2.

Graph (T) is closed:

$\shortparallel$

$\{(f, gf) : f \in \mathcal{D}(M_g)\}$, suppose $(f_n, gf_n) \xrightarrow[\text{in } L^2 \times L^2]{} (f, h) \in L^2 \times L^2$

need to show that $f \in \mathcal{D}(M_g)$ & $h = gf$.

Pf in the case where $|g(x)| \le 1$:

In this case M_g is cts, $f_n \to f \Rightarrow g f_n \to gf \Rightarrow gf = h$

Pf in the case where $|g(x)| \ge 1$

$g f_n \to h$ we know here that $M_{\frac{1}{g}}$ is a bounded operator

$$\left.\begin{array}{l} \Rightarrow M_{\frac{1}{g}}(g f_n) = f_n \to M_{\frac{1}{g}}(h) = \frac{1}{g} h \\ f_n \to f \end{array}\right\} \Rightarrow h = gf.$$

Last case

Write $X = X_{\le 1} \cup \underbrace{X_{>1}}_{= \{x \in X \mid |g(x)| > 1\}}$

$L = \{x \in X \mid |g(x)| \le 1\}$

Projection-operators: $P_{\le 1} : L^2(X, \mu) \to L^2(X_{\le 1}, \mu)$

$f \in L^2 \to f_{\le 1}$

If $(f_n, g f_n) \to (f, h) \Rightarrow (f_n)_{\le 1} \to (f)_{\le 1}$

$g(f_n)_{\le 1} \to (h)_{\le 1}$

$$\left.\begin{array}{l} \text{on } X_{\le 1} \text{ we have } g(f_{\le 1}) = h_{\le 1} \\ \text{Similarly: } g(f_{>1}) = (h)_{>1} \end{array}\right\} \Rightarrow gf = h.$$

Moreover, $(M_g)^* = M_{\overline{g}}$ & $\mathcal{D}(M_g^*) = \mathcal{D}(M_g)$

Pf: Suppose $f, h \in \mathcal{D}(M_g)$

$$\langle M_g f, h \rangle = \langle f, M_{\overline{g}} h \rangle$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$\int_X g f \overline{h} \, d\mu = \int_X f \overline{\overline{g} h} \, d\mu$$

Therefore, M_g is self-adjoint if $g(x) \in \mathbb{R}$ $\forall x \in X$

<u>Thm</u>: If T from H to H is a self-adjoint operator (closed), then one can split H into a countable orthogonal sum of closed subspaces H_n $(H \cong \bigoplus_n H_n)$ such that $H_n \subseteq \mathcal{D}(T)$, $T|_{H_n}$ is a bounded operator and $T(H_n) \subset H_n$ (s.t. $T|_{H_n} : H_n \to H_2$ has spectrum in $[-n,-n+1] \cup [n-1,n]$)

<u>Corollary</u> (spectral theory of unbdd operators): H is separable. T as above, then there exists a σ-finite measure space (X,μ) and a measurable function $g: X \to \mathbb{R}$ s.t. H is isomorphic to $L^2(X,\mu)$ via a unitary map $\phi : H \to L^2(X,\mu)$ and

$$\phi(Tv) = M_g \phi(v) \quad \text{for } v \in \mathcal{D}(T)$$

$$\phi(\mathcal{D}(T)) = \mathcal{D}(M_g).$$

Pf: Let $T_n = T|_{H_n} : H_n \to H_n$, then T_n is self-adjoint: $v,w \in H_n$, $\langle Tv, w\rangle = \langle v, Tw\rangle$ since $H_n \subseteq \mathcal{D}(T) = \mathcal{D}(T^*)$
$T = T^*$.

Hence $\exists \phi_n : H_n \to L^2(X_n, \mu_n)$ & $g_n : X_n \to \mathbb{R}$

$$\phi_n(Tv) = g_n \phi_n(v) \quad \text{for } v \in H_n$$

This holds for all n, Now define

$$X = \bigsqcup_n X_n \ , \quad \mu(A) := \sum_{n \geq 1} \mu_n(A \cap X_n)$$

& μ is a σ-finite measure.

$L^2(X,\mu) \supset L^2(X_n,\mu_n)$ since a function on X_n can be extended to X by setting it $\equiv 0$ on $X \setminus X_n$.

$$L^2(X,\mu) = \bigoplus_{n=1}^{\infty} L^2(X_n,\mu_n)$$ by some version of Parseval.

$$\phi \uparrow \qquad\qquad \uparrow \phi_n$$

$$H = \bigoplus_{n=1}^{\infty} H_n.$$

$$\phi(v) = \sum_{n=1}^{\infty} \phi_n(v_n) \quad \text{where } v_n \in H_n \ \& \ v = \sum v_n$$

By Thm / construction of ϕ_n:

$\phi(T(v_n)) = M_g\,\phi(v_n)$ for $g(x) = g_n(x)$ for $x \in X_n$.
for $v_n \in H_n$

If $v \in \mathcal{D}(T)$ then $v = \sum_n v_n$, $v_n \in H_n$

$\Rightarrow \phi(T_n) = M_g\,\phi(v_n)$ for all n.

$T(v_n) \in H_n$

Claim: $T(\sum_m v_m) = \sum_m T v_m$

Pf: Let $w \in H_n$ & $v = \sum_m v_m \in \mathcal{D}(T)$

$$\langle Tv, w\rangle = \langle v, \underbrace{Tw}_{\in H_n}\rangle = \langle v_n, Tw\rangle = \langle Tv_n, w\rangle$$

$Tv = \sum_m v_m'$ because $H = \bigoplus H_m$

$$\underset{w \in H_n}{\langle Tv, w\rangle} = \langle v_n', w\rangle \Rightarrow v_n' = Tv_n$$

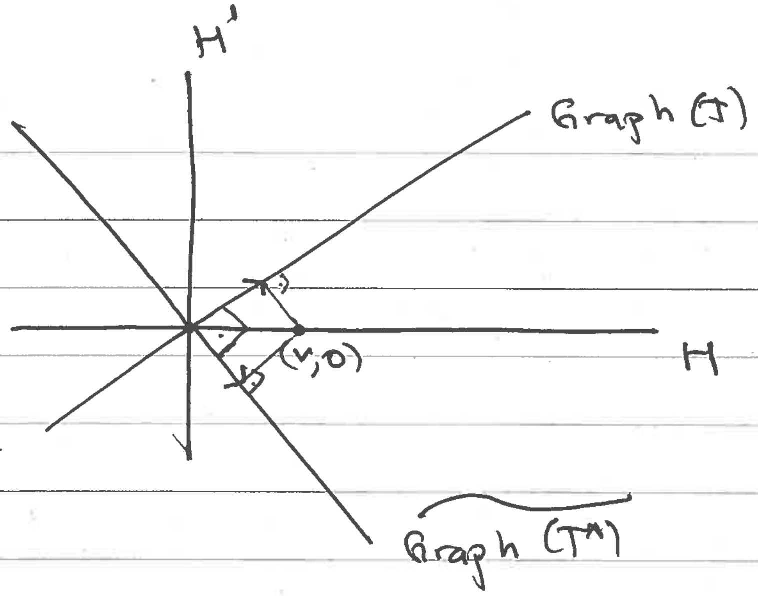
Thus

$$\phi(Tv) = \phi\left(\sum_n T_n\right) = \sum_n \phi_n(Tv_n) = \sum_n M_{g_n}\phi_n(v_n)$$

$$= M_g\left(\phi\left(\sum_n v_n\right)\right) = M_g(\phi(v)) \qquad \square$$

<u>Pf of Thm for the operator T^*T, T closed with dense domain:</u>

Assume T from H to H' a closed operator with dense domain. This defines T^* - we will prove the conclusions of thm for T^*T:



As Graph (T) & $\overline{\text{Graph}(T^*)}$ are orthogonal complements

we have $(v,0) = (w, Tw) \oplus (-T^*w_1, w_1)$

$w \in \mathcal{D}(T) \subset H$

$w' \in \mathcal{D}(T^*) \subset H'$

$0 = Tw + w_1$

$v = w - T^*w_1 = w + T^*Tw = (I + T^*T)w$

Define $B(v) = w$; $B = P_{H\times H' \to H} \underset{H\times H' \to H\times H'}{P_{\text{Graph}(T)}} \iota_{H \to H\times H'}$

- Adj on orth proj is itself.
- Adj of embedding is an orth proj.

$\Rightarrow B^* = B$ bdd.

$$I = (I + T^*T)B$$

, 0 is in spec $\Leftrightarrow$ T^*T is unbdd

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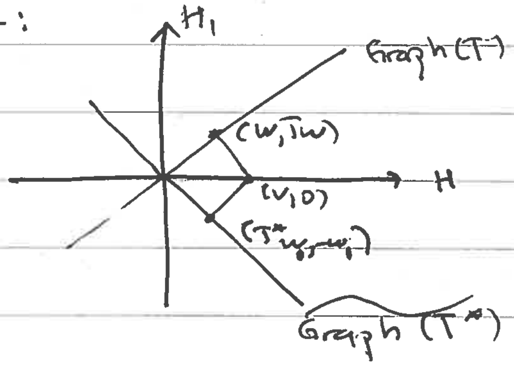
Thm: H, H' Hilbert space. T from H to H' a closed operator with dense domain. Then T^*T from H to H is isomorphic to a multiplication operator ~~on~~ M_g on $L^2(X,\mu)$ for a σ-finite measure space (X,μ) and a measurable $g \geq 0$.

Pf:



$$(v,0) = (w, Tw) + (T^*w_1, -w_1) \Rightarrow w_1 = Tw, \quad w \in \mathcal{D}(T)$$
$$w_1 \in \mathcal{D}(T^*)$$

$$\Rightarrow w \in \mathcal{D}(T^*T)$$

Also, $v = w + T^*w_1 = w + T^*Tw$

we define $B(v) = w$, obviously B is linear.

$$\underset{\underset{H}{\cap}}{v} \xmapsto{\iota_H} \underset{\in_{H\times H'}}{(v,0)} \xrightarrow[\substack{\uparrow \\ \text{orth. proj.}}]{P_{\text{Graph}T}} (w, Tw) \xmapsto{P_H} w$$

$$B = P_H \circ P_{\text{Graph}(T)} \circ \iota_{H \to H\times H'}$$

$$B^* = \underset{\underset{P_{H\times H' \to H}}{\parallel}}{\iota^*_{H\to H\times H'}} \circ \underset{\underset{P_{\text{Graph}(T)}}{\parallel}}{P^*_{\text{Graph}(T)}} \circ \underset{\underset{\iota_{H\to H\times H'}}{\parallel}}{P^*_{H\times H' \to H}} = B$$

$\therefore \quad B^* = B$, B is selfadj and bdd.

$$v = (I + T^*T)Bv \quad \forall v \in H \Rightarrow (I + T^*T)B = I$$

i.e. $I+T^*T$ is a left-inverse to B.

[This is not the same as saying $0 \notin \sigma(B)$ (since we do not know if B has also a right inverse and that its inverse is bdd)..]

This implies B is injective. We know $H \circlearrowleft B \cong L^2(X,\mu) \circlearrowleft M_h$

So, $h \neq 0$ a.e. (b/c if $A = \{x \in X \mid h(x) = 0\}$ has pos. measure, then $\forall B \subseteq A$ with $\mu(B) < \infty$, $\mu(B) > 0$, we would have $M_h(1_B) = 0$, $1_B \in L^2(X,\mu)$.

a contradiction to injectivity of M_h, B).

<u>Claim</u>: $\exists$ subspaces H_n closed, orthogonal to each other s.t.

$H = \bigoplus_n H_n$, $B: H_n \to H_n$ and $\sigma(B|_{H_n}) \subseteq [\frac{1}{n+1}, \frac{1}{n}]$

<u>Pf</u>:

(1) $\|B\| \leq 1 \Rightarrow \sigma(B) \subseteq [-1,1]$.

(2) $\langle \underbrace{Bv}_{w}, v \rangle = \langle w, (I+T^*T)w \rangle = \langle w,w \rangle + \langle w, T^*Tw \rangle$
$= \langle w,w \rangle + \langle Tw, Tw \rangle \geq 0$.

So, B is positive and hence $\sigma(B) \subseteq [0,1]$

and $h: X \to (0,1]$

$H_n =$ pre-image of $L^2(X_n, \mu|_{X_n}) \subseteq L^2(X,\mu)$ where

$X_n = \{x \in X \mid h(x) \in (\frac{1}{n+1}, \frac{1}{n}]\}$.

then $X = \bigsqcup_n^\infty X_n$, $L^2(X,\mu) = \bigoplus_n^\infty L^2(X_n, \mu|_{X_n})$.

Hence $\exists$ an operator $C_n: H_n \to H_n$ bounded which is the inverse of $B|_{H_n}$. ($C_n \cong M_{h^{-1}}$ on X_n)

$C_n(v) = (I+T^*T)BC_n(v) = (I+T^*T)v$ $\therefore$ $C_n = I+T^*T$ on H_n.

We know from H to H. operator $I+T^*T$.

from $L^2(X,\mu)\to L^2(X,\mu)$ operator $M_{h^{-1}}$ self-adj., dense domain.

& $I+T^*T \cong M_{h^{-1}}$ on $H_n \cong L^2(X_n, \mu_n|_{X_n})$

<u>Claim</u>: Graph $(I+T^*T)$ is closed.

$$\mathrm{Graph}(B) \subseteq \begin{array}{c}\mathrm{Graph}(I+T^*T)\\ \cap \\ H\times H\end{array}$$

Is it true that Graph $(I+T^*T) = \{(Bv,v) : v\in H\}$

We know already that $\{(Bv,v) \mid v\in H\} \subset \mathrm{Graph}(I+T^*T)$

Suppose $w\in \mathcal{D}(I+T^*T)$, $(w, \underbrace{(I+T^*T)w}_{\in H}) \perp (Bv,v)$ $\forall v\in H$

let $v=(I+T^*T)w$

$\langle w, Bv\rangle + \langle v,v\rangle = 0$

$-\langle v,v\rangle = \langle Bw, v\rangle = \langle Bw, (I+T^*T)w\rangle = \langle Bw,w\rangle + \langle T^*TBw,w\rangle$

$= \langle (I+T^*T)Bw,w\rangle = \langle w,w\rangle$ $\times$ $\Rightarrow w=0$

$\Rightarrow$ Graph $(I+T^*T)$ is a function of the graph of B.

In the same way Graph $(M_{h^{-1}})$ is the same function of Graph (M_h)

Graph (T^*T) is a function of Graph $(I+T^*T)$

$$(\bar{h}, \bar{h}^{1}-\bar{h}) \longleftarrow (\bar{h}, \bar{h}^{1})$$

$$(v, T^*Tv) \longleftarrow (v, (I+T^*T)v)$$

The operator T^*T from H to H is isomorphic to the operator $M_{h^{-1}-1}$ from $L^2(X,\mu)$ to $L^2(X,\mu)$.

The isomorphism $\phi: H\to L^2(X,\mu)$ coming from the spectral thm for B.