D-MATH NASPDE FS 2010
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Problem Set 8

1. Let H be a separable Hilbert space and let N_Q be a nondegenerate Gaussian measure on H. The Cameron–Martin space $Q^{1/2}(H) \subset H$ is a separable Hilbert space when endowed with the scalar product

$$\langle z, w \rangle_{Q^{1/2}(H)} := \langle Q^{-1/2} z, Q^{-1/2} w \rangle_H \ , \quad z, w \in Q^{1/2}(H) \ .$$

a) Let $z_1, \dots, z_n \in Q^{1/2}(H)$. Show that the white noise maps $(W_{Q^{-1/2} z_i})_{i=1}^n$ are independent if and only if $(z_i)_{i=1}^n$ are mutually orthogonal in $Q^{1/2}(H)$.

Hint: Use Proposition II.5.5.ii).

b) Show that for any $z \in Q(H)$,

$$W_{Q^{-1/2} z}(x) = \langle z, x \rangle_{Q^{1/2}(H)} \qquad \forall x \in Q^{1/2}(H) \ .$$

In particular, if $(z_m)_{m \in \mathbb{N}} \subset Q(H)$ is an orthonormal basis of $Q^{1/2}(H)$, then

$$x = \sum_{m=1}^{\infty} W_{Q^{-1/2} z_m}(x) z_m \qquad \forall x \in Q^{1/2}(H) \ ,$$

with convergence in $Q^{1/2}(H)$, and the coefficients $W_{Q^{-1/2} z_m}(x)$ in this series are independent. One can show that this equation holds with convergence in H for N_Q-a.e. $x \in H$ and without the restriction $z_m \in Q(H)$.

c) Derive the Karhunen–Loève series as a special case of the above series expansion in H.

Hint: Define an orthonormal basis of $Q^{1/2}(H)$ using the eigenvectors e_m of Q.

2. Consider the differential equation

$$-\frac{d}{dx}(1 + 0.85 y x \cos(3\pi x) \frac{d}{dx} u(y, x)) = 1 \ , \quad x \in (0, 1) \ ,$$

with homogeneous Dirichlet boundary conditions $u(y, 0) = u(y, 1) = 0$ and a stochastic parameter y uniformly distributed on $[-1, 1]$. We interpret the left hand side as a sum of a deterministic operator

$$A := -\frac{d^2}{dx^2} : H_0^1(0, 1) \to H^{-1}(0, 1) \ ,$$

and a perturbation

$$R := -\frac{d}{dx}(0.85 x \cos(3\pi x) \frac{d}{dx}) \ .$$

Furthermore, we set $f(x) := 1$ such that the stochastic differential equation becomes

$$(A + yR)u = f \ , \quad y \in [-1, 1] \ .$$

Bitte wenden!

Let $(L_k)_{k=0}^{\infty}$ be Legendre polynomials on $[-1,1]$, scaled to be an orthonormal basis of $L^2([-1,1],\frac{1}{2}dy)$. They satisfy the three term recursion

$$\beta_{k+1}L_{k+1}(y) = yL_k(y) - \beta_k L_{k-1}(y)\ , \quad \beta_k = \frac{1}{\sqrt{4-k^{-2}}}\ , \quad k \in \mathbb{N}\ ,$$

with $L_{-1} := 0$ and $L_0 := 1$. Then the solution u can be expanded as

$$u(y,x) = \sum_{k=0}^{\infty} u_k(x)L_k(y)$$

with coefficients $u_k \in H_0^1(0,1)$.

To discretize the stochastic differential equaition, we truncate this series, and approximate the coefficients u_k by linear finite elements. This results in a finite dimensional linear system, which we solve by a preconditioned conjugate gradient iteration.

a) Use the routines `stiff_gen` and `load_ellip` to assemble matrices for finite element approximations of A, R and f.

Hint: Note that these functions require that the vertices of the finite element mesh are entered as a column vector. Also, remember to remove the boundary degrees of freedom.

b) Assume that the (approximate) Legendre coefficients u_k of u are stored as the columns of a matrix. Write a function that, given these coefficients, computes the coefficients of $(A + yR)u$.

Hint: Solve for $yL_k(y)$ in the three term recursion for the Legendre polynomials.

c) Solve the discretized stochastic differential equation using the preconditioned conjugate gradient routine `pcgsolve`. The first argument is the right hand side to the linear system, which is the Legendre coefficients of f, *i.e.* a matrix of the appropriate size with the first column equal to coefficients of f and all others equal to zero. The second argument of `pcgsolve` is a function handle to a routine that applies the discretized operator $A + yR$ to a vector, *i.e.* a matrix containing the Legendre coefficients of a stochastic vector in its columns, taking only the vector as an argument. Use the routine from the previous subtask. The third argument of `pcgsolve` is a function handle to a preconditioner. Use a solve by the backslash routine of the deterministic equation $Au = f$. Plot the decay of the energy norm (A-norm) of the coefficients of the solution.

d) Compute the mean of the solution u, without evaluating any Legendre polynomials. Plot it together with the solution of $Au = f$.

Hint: Since $(L_k)_{k=0}^{\infty}$ are orthogonal in $L^2([-1,1],\frac{1}{2}dy)$, all L_k with $k \geq 1$ have zero mean.

e) Derive a representation for the covariance of u using only the Legendre coefficients u_k. Plot the covariance kernel on $(0,1)^2$.

Deadline: Wednesday, June 2, 2010
Webpage: www.math.ethz.ch/education/bachelor/lectures/fs2010/math/numsol
Coordinator: Claude Gittelson, HG G 56.1, claude.gittelson@sam.math.ethz.ch
Testat requirement: reasonable solution to 70% of the exercises

Theoretical and computational Master and Ph.D. theses on numerical analysis of stochastic PDE are available. Please contact Prof. Ch. Schwab or Claude Gittelson if you are interested.