Christian Engström
Carlos Jerez-Hanckes
Daniel Kressner

Exercise VII: The DtN map

1. Scattering problems in the upper half-plane

Let $\mathbb{R}^2_+$ denote the upper half-plane. Consider an impinging plane-wave u^{inc} with wave-vector $\mathbf{k} = (k\cos\vartheta, k\sin\vartheta)$ scattered from a bounded object $\Omega \in \mathbb{R}^2_+$ whose closed boundary is Γ_1. Let Γ_0 denote the line given by the condition $x_2 = 0$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and introduce the complement domain $\Omega^c := \mathbb{R}^2_+ \setminus \bar{\Omega}$. The total wave u^{tot} can described as the sum

$$u^{\text{tot}} = u^{\text{inc}} + u \tag{1}$$

where u is the scattered wave. The total wave u^{tot} and the impinging plane wave u^{inc} satisfies the homogeneous Helmholtz-type equation

$$-\Delta u(\mathbf{x}) - k^2 u(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Omega^c. \tag{2}$$

The above is problem is not well-defined unless boundary conditions and radiation conditions for the scattered field are included.

a) State the correct radiation conditions for the problem.

b) Study (2) with homogeneous Dirichlet condition on Γ_0 and on Γ_1:

1. Use the method of images to find the Green's function.
2. Write solution of the homogeneous equation using Hankel series.
3. Find the exact DtN map at a suitable artificial boundary Γ_R.

c) Study (2) with homogeneous Neumann condition on Γ_0 and on Γ_1:

1. Use the method of images to find the Green's function.
2. Write solution of the homogeneous equation using Hankel series.
3. Find the exact DtN map at a suitable artificial boundary Γ_R.

Solutions due on Tuesday 18th, May 2010