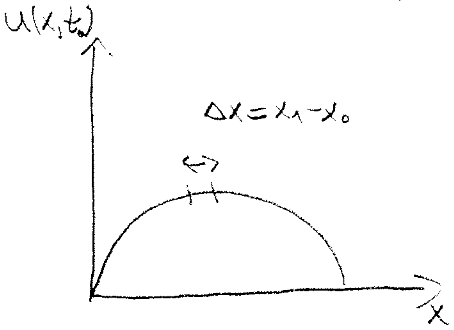
(1) The violin string (wave equation)

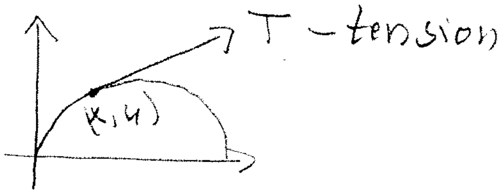


$\rho(x)$ – density
$m = \rho \Delta x$ – mass
u_{tt} – acceleration

Newton's law

$ma = f$ – force

The force term



Assume
* no gravitation (low mass)
* tension in direction of the tangent $\hat{t}$ (the string do not resist being bent)

Tangent at $x = x_0$ (fix t)

$u = u(x_0) + u_x(x_0)(x - x_0)$, $\hat{t} = \frac{1}{\sqrt{1+u_x^2}}(1, u_x)$

$T = A\hat{t}$, A – amplitude (the tension)

Assume: $T(x) = T$; $\rho(x) = \rho$

$\boxed{\text{Vertical acceleration } u_{tt}} = \boxed{\text{Vertical part } f \text{ of } T}$

$$f = A \frac{u_x}{\sqrt{1+u_x^2}} \Bigg|_{x=x_0}^{x=x_1}$$

$\Rightarrow$

$$\rho u_{tt} = \underbrace{A \frac{1}{\Delta x} \frac{u_x}{\sqrt{1+u_x^2}} \Bigg|_{x=x_0}^{x=x_1}}_{= \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)}$$ in the limit $\Delta x = x_1 - x_0 \to 0$

The nonlinear wave eq.

$$\rho u_{tt} - A \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right) = 0$$

ca 20 min

The linear wave eq

Assume small oscillations: $u_x^2 < u_x < 1$

$$\Rightarrow \quad \frac{u_x}{\sqrt{1+u_x^2}} = u_x + O(u_x^3) \approx u_x$$

$$\boxed{u_{tt} - c^2 u_{xx} = 0 \, , \quad c = \frac{A}{\varphi}}$$

General solution

Let $\xi = x - ct$, $\eta = x + ct$

$\Rightarrow$

$$u_x = u_\xi \cdot 1 + u_\eta \cdot 1 \quad ; \quad u_t = u_\xi(-c) + u_\eta c$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \quad ; \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

$$0 = u_{tt} - c^2 u_{xx} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta})$$
$$= -4c^2 u_{\xi\eta}$$

$$\Rightarrow \quad u = \phi(\xi) + \psi(\eta)$$

$$\boxed{u(x,t) = \phi(x - ct) + \psi(x + ct)}$$

right going wave (outgoing) — left going wave (incoming)

(3) Time harmonic waves

We want to specify an incoming wave with a fixed frequency and not the data $u(x,0)$ & $u_t(x,0)$

Assume time-harmonic waves:

$U(x,t) = u(x) e^{-i\omega t}$ & $U_{tt} - c^2 U_{xx} = 0$

$\Rightarrow$ $-\omega^2 u - c^2 u'' = 0 \iff \boxed{u'' + k^2 u = 0, \; k = \frac{\omega}{c}}$ Helmholtz eq.

ω – (angular) frequency ; c – wave speed ; k – wave number

Plane waves

$u(x) = Ae^{ikx} + Be^{-ikx}$

$u(x+\lambda) = u(x)$ when $\lambda = \frac{2\pi}{k}$

λ – wave length

$U(x,t) = Ae^{i(kx-\omega t)} + Be^{i(kx-\omega t)}$ – plane wave(s)

constant value if the differential is zero

$v_p = \frac{dx}{dt} = \frac{\omega}{k}$ – phase velocity

e^{ikx} – right going or $\boxed{\text{outgoing}}$ with $v_p = c$

e^{-ikx} – left going or $\boxed{\text{incoming}}$ with $v_p = -c$

We must eliminate the incoming wave to get an unique solution!

Non-reflecting boundary conditions

Take $u' - iku = 0$ at $x = x_0$

$\Rightarrow$ eliminate e^{-ikx} from the set $\{e^{ikx}, e^{-ikx}\}$

This do only work in 1D!

(4) Helmholtz eq. in 3D

$$\Delta u + k^2 u = 0, \quad x \in \mathbb{R}^3 \setminus \bar{\Omega} =: \Omega^+$$

$$\boxed{\text{No simple non-reflecting boundary condition exist when dim} > 1}$$

Sommerfeld's radiation condition

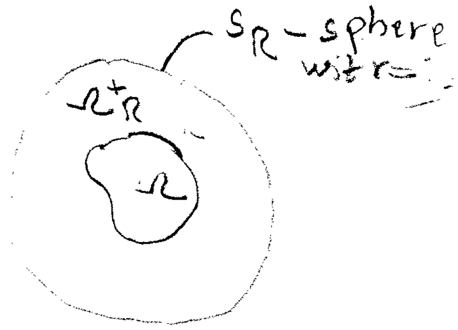
Fundamental solution

$$\Delta G + k^2 G = \delta(x) \Rightarrow G(|x'-x|) = \frac{e^{ik|x'-x|}}{4\pi |x'-x|}$$

Green's theorem $\Rightarrow$

$$u(x) = \int_{\partial\Omega} \underbrace{\left\{ u(x') \frac{\partial}{\partial n(x')} G(x,x') - G(x,x') \frac{\partial}{\partial n(x')} u(x') \right\}}_{F(x,x')} dS(x')$$

Truncate the domain:



$\Omega^+_R \to \Omega^+, \quad R \to \infty, \quad \partial\Omega^+_R = \partial\Omega \cup S_R$

Demand: $\int_{S_R} F(x,x')\, dS(x') \to 0, \quad R \to \infty$

Fix $x \in \Omega^+$: We have $|x'-x| \approx |x'|$ and $\frac{\partial}{\partial n} = \frac{\partial}{\partial R}$ when R large and $x' \in S_R$

$$\Rightarrow \int_{S_R} \frac{1}{R}\left(iku - \frac{u}{R} - \frac{du}{dR}\right) \frac{e^{ikR}}{4\pi} \underbrace{dS}_{\sim R^2} \to 0, \quad R \to \infty$$

"$\Rightarrow$" (a) $Ru < \infty$ for all R

(b) $R\left(iku - \frac{du}{dR}\right) \to 0, \quad R \to \infty$

(i) It can be shown that (b) $\to$ (a)

(ii) (b) is the Sommerfeld condition