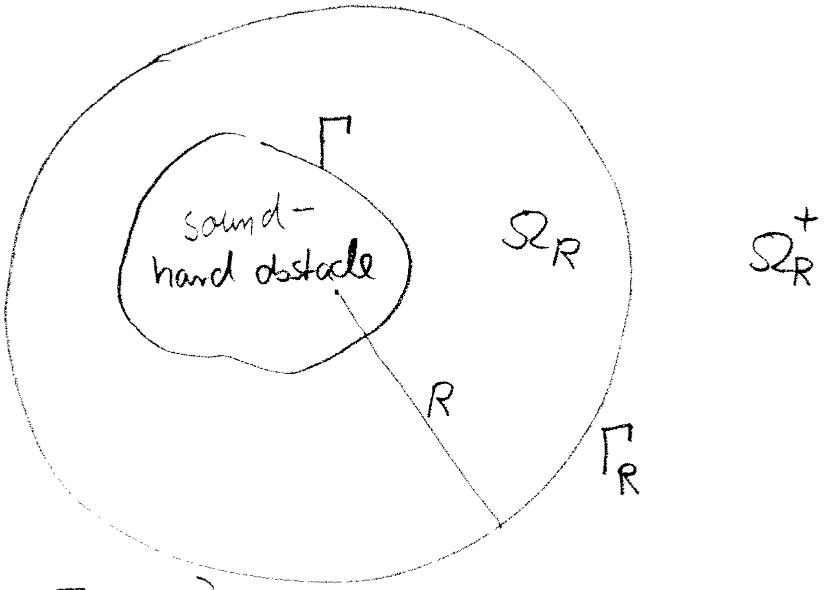
The Finite Element Method in the Near Field



$$\left.\begin{aligned} \frac{\partial u_-}{\partial n} &= g \quad \text{on } \Gamma \\ -\Delta u_- - k^2 u_- &= 0 \quad \text{in } \Omega_R \end{aligned}\right\} \text{interior problem}$$

$$\left.\begin{aligned} u_- &= u_+ \quad \text{on } \Gamma_R \\ \frac{\partial u_-}{\partial n} &= \frac{\partial u_+}{\partial n} \end{aligned}\right\} \text{coupling on artificial boundary}$$

$$\left.\begin{aligned} -\Delta u_+ - k^2 u_+ &= 0 \quad \text{in } \Omega_R^+ \\ \lim_{r\to\infty} r^{(d-1)/2}\left(\frac{\partial u_+}{\partial n} - iku_+\right) &= 0 \end{aligned}\right\} \text{exterior problem}$$

$$\Big\Downarrow \quad \text{DtN map } M \text{ on } \Gamma_R$$

$$\left.\begin{aligned} \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma \\ -\Delta u - k^2 u &= 0 \quad \text{in } \Omega_R \\ \frac{\partial u}{\partial n} &= Mu \end{aligned}\right\} \text{reduced problem on bounded domain}$$

Weak formulation of reduced problem: ②

Find $u \in H^1(\Omega_R)$ such that $\forall v \in H^1(\Omega_R)$

$$(*) \qquad \underbrace{\int_{\Omega_R} (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v})\, dx - \int_{\Gamma_R} M u \bar{v}\, dS}_{=: a(u,v)} = \underbrace{\int_{\Gamma} g \bar{v}\, dS}_{=: f(v)}$$

For numerical computation: $M \rightsquigarrow \tilde{M}$ (truncated DtN, local ABC, ...). We do not worry about this issue too much at the moment and consider discretization of $(*)$.

Δ-inequality: total error $\leq$ discretization error + b.c. error

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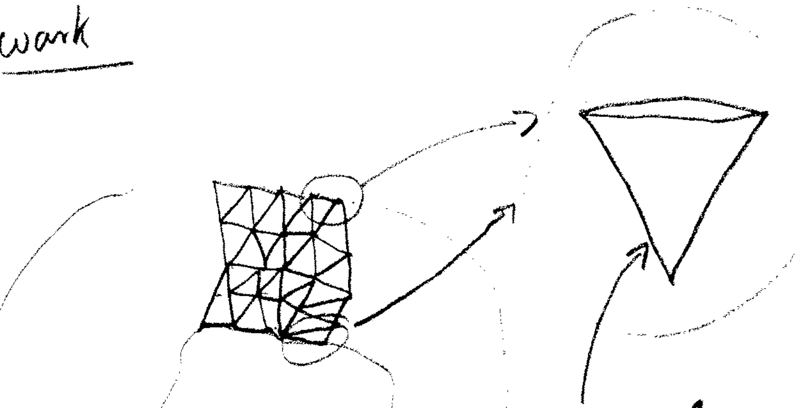
Finite Element Framework

Ingredient 1: Mesh $\mathcal{M}$



For example, triangulation:

$\mathcal{M} = \{K_i\}$, $K_i \hat{=}$ open triangle

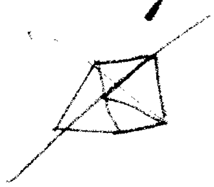
No overlap: $K_i \cap K_j = \emptyset$, $i \neq j$

$$\bigcup_{i=1}^{\#\mathcal{M}} \overline{K_i} = \overline{\Omega_R}$$

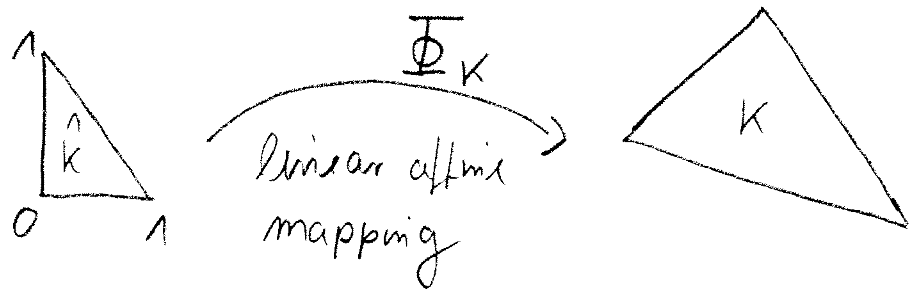
$\overline{K_i} \cap \overline{K_j}$, $i \neq j$ $\nearrow \emptyset$ / $\rightarrow$ edge of both triangles / $\searrow$ vertex of both triangles

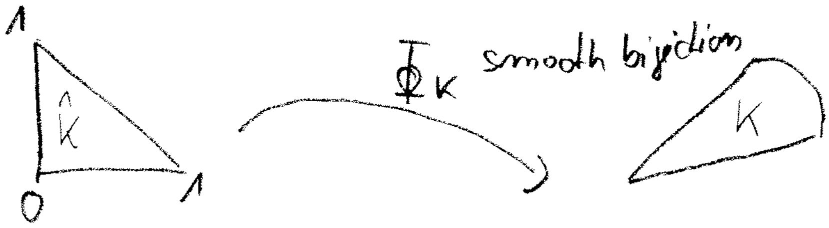
triangle (OK for linear finite elements) or curvilinear triangle (needed for higher order elements)

$\Rightarrow$ $\boxed{\text{No hanging nodes:}}$



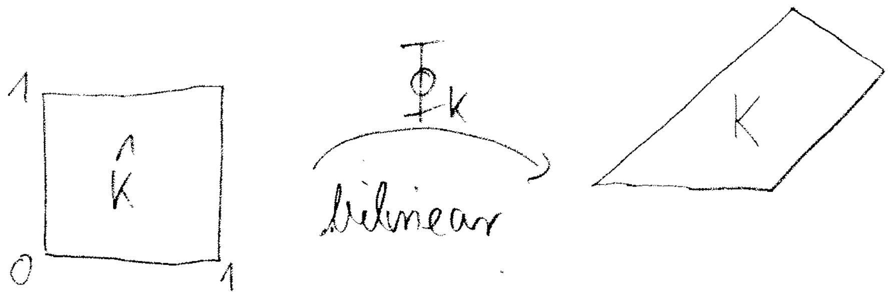
Construction from Master element

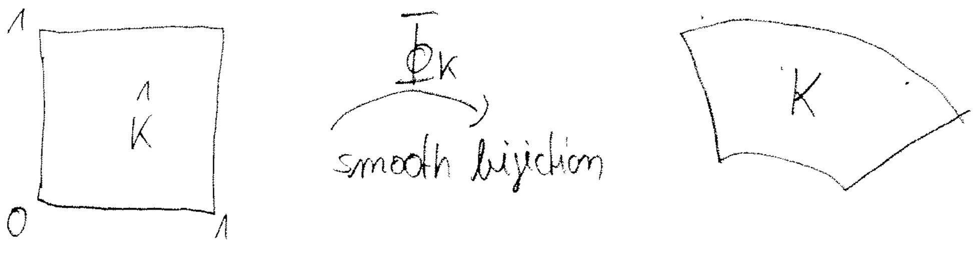




$J_K \hat{=}$ Jacobian matrix of Φ_K

Quadrilaterals





(Curvilinear) Triangles + Quadrilaterals can be mixed in $\mathcal{M}$

3D: Tetrahedra + Bricks

Ingredient 2: Finite element space

$$S_p(\mathcal{M}) = \{ v : v|_K \in P_p(K) \quad \forall K \in \mathcal{M} \}$$

$\underline{d=2}$

$p=1: \quad v(x_1, x_2) = \alpha_{10} x_1 + \alpha_{01} x_2 + \alpha_{00}$

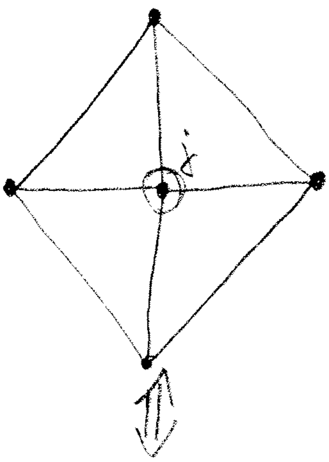
$p=2: \quad v(x_1, x_2) = \alpha_{20} x_1^2 + \alpha_{11} x_1 x_2 + \alpha_{02} x_2^2 + \alpha_{10} x_1 + \alpha_{01} x_2 + \alpha_{00}$

$\Rightarrow$ sum of monomial degrees $\leq p$

Suitable for triangles, tetrahedra, ...

Global shape functions

$p=1$:

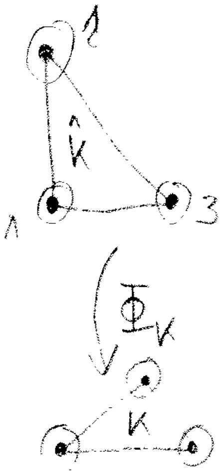


At node x^j: $b_j(x^j) = 1$

$b_j(x^i) = 0 \quad i \neq j$

$\rightsquigarrow b_1, \dots, b_N$ basis of $S_1(\mathcal{M})$ — N = number of nodes in $\mathcal{M}$

Local shape functions



$b_1^{\hat{K}} = 1 - \hat{x}_1 - \hat{x}_2$

$b_2^{\hat{K}} = \hat{x}_2$

$b_3^{\hat{K}} = \hat{x}_1$

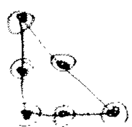
$\xrightarrow{\Phi_K}$ $b_1^K(\Phi_K \hat{x}) = b_1^{\hat{K}}(\hat{x})$

$b_2^K(\Phi_K \hat{x}) = b_2^{\hat{K}}(\hat{x})$

$b_3^K(\Phi_K \hat{x}) = b_3^{\hat{K}}(\hat{x})$

Construction of global shape functions by gluing local shape functions of adjacent triangles together

$p=2$:



See lecture notes by R. Hiptmair for pictures.

$$S_p(\mathcal{M}) = \{v : v|_K \in Q_p(K) \quad \forall K \in \mathcal{M}\}$$

$\underline{d=2}$ $\quad p=1$: $\quad v(x_1, x_2) \in \mathrm{span}\{1, x_1, x_2, x_1 x_2\}$

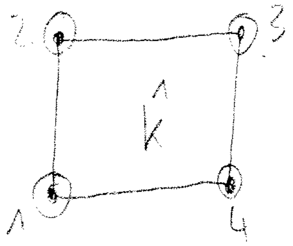
$p=2$: $\quad v(x_1, x_2) \in \mathrm{span}\{1, x_1, x_2, x_1 x_2, x_1 x_2^2, x_1^2 x_2, x_1^2 x_2^2\}$

$\Rightarrow$ maximal monomial degree $\leq p$.

Suitable for quadrilaterals, bricks, ...

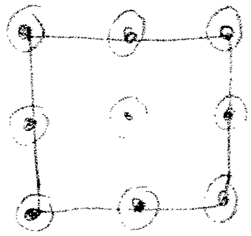
Local shape functions

$\underline{p=1}$



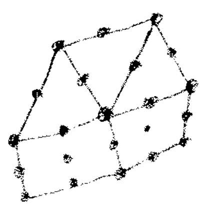
$$b_1^{\hat{K}} = (1-\hat{x}_1)(1-\hat{x}_2)$$
$$b_2^{\hat{K}} = (1-\hat{x}_1)\hat{x}_2$$
$$b_3^{\hat{K}} = \hat{x}_1 \hat{x}_2$$
$$b_4^{\hat{K}} = \hat{x}_1(1-\hat{x}_2)$$

$\underline{p=2}$



See R. Hiptmair's lecture notes

Triangles + quadrilaterals can be mixed but local degrees of freedom must match:



Galerkin discretization

$$V_N = S_p(\mathcal{M}) \subseteq V$$

Example: V_N = piecewise linear, $V = H^1$

Linear variational problem:

Find $u \in V$ s.t. $a(u,v) = f(v) \quad \forall v \in V. \quad (*)$

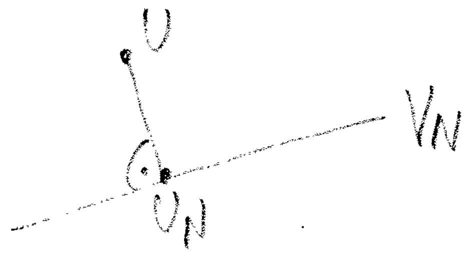
Discretized linear variational problem

Find $u_N \in V_N$ s.t. $a(u_N, v_N) = f(v_N) \quad \forall v_N \in V_N \quad (*')$

Remark: V-ellipticity of a $\Rightarrow$ V_N-ellipticity of a (same constant)

$(*) + (*') \Rightarrow$ Galerkin-orthogonality:

$$a(u - u_N, v_N) = 0 \quad \forall v_N \in V_N \qquad (GO)$$



Assume: Discrete inf-sup conditions are satisfied

(i) $$\beta_N = \inf_{\substack{u_N \in V_N \\ u_N \neq 0}} \sup_{\substack{v_N \in V_N \\ v_N \neq 0}} \frac{|a(u_N, v_N)|}{\|u_N\|_V \|v_N\|_V} > 0$$

(ii) $$\sup_{u_N \in V_N, u_N \neq 0} |a(u_N, v_N)| > 0 \quad \forall v_N \in V_N, v_N \neq 0$$

(note: since V_N is finite-dim. (i) $\Rightarrow$ (ii))

Then

$$\|u - u_N\|_V \leq \left(1 + \frac{\|a\|}{\beta_N}\right) \inf_{v_N \in V_N} \|u - v_N\|_V \quad (QO)$$

Proof:

(i) $\Rightarrow$

$$\beta_N \| u_N - v_N \|_V \leq \sup_{w_N \in V_N} \frac{|a(u_N - v_N, w_N)|}{\| w_N \|}$$

$$\overset{(60)}{=} \sup_{w_N} \frac{|a(u - v_N, w_N)|}{\| w_N \|} \leq \|a\| \, \| u - v_N \|$$

$$\| u - u_N \| = \| u - v_N + v_N - u_N \|$$

$$\leq \left(1 + \frac{\|a\|}{\beta_N} \right) \| u - v_N \|$$

□

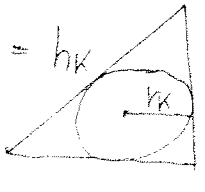
Meaning of QO (Quasi-Optimality): If $\frac{\|a\|}{\beta_N}$ does not deteriorate $\Rightarrow$ Solution of (*') nearly recovers best. approx. of u contained in V_N.

$$\text{Convergence} = \underbrace{\text{Stability}}_{\frac{\|a\|}{\beta_N}} \times \text{Approximation/Consistency}$$

Typical best approximation results

Mesh geometry parameters

diameter = h_K



$\Rightarrow \frac{h_K}{r_K} = \rho_K$ = shape regularity of K

$h_{\mathcal{M}} = \max_{K \in \mathcal{M}} h_K$ = meshwidth

$\rho_{\mathcal{M}} = \max_{K \in \mathcal{M}} \rho_K$ = shape regularity

Assume $u \in H^2$, $V_N = S_1(\mathcal{M})$

$$|u - I_p u|_{H^1(\mathcal{M})} \leq C\, h_{\mathcal{M}}\, |u|_{H^2(\Omega)}$$

I_p: nodal interpolation

C: depends on $\rho_{\mathcal{M}}$

$$\|u - I_p u\|_{L^2(\mathcal{M})} \leq C\, h_{\mathcal{M}}^2\, |u|_{H^2(\Omega)}$$

More general:

$u \in H^k(\Omega)$, $V_N = S_p(\mathcal{M})$, $\boxed{2 \leq k \leq p+1}$

$$|u - I_p u|_{H^m(\mathcal{M})} \leq C\, h_{\mathcal{M}}^{k-m}\, |u|_{H^m(\Omega)}$$

C: depends on $\rho_{\mathcal{M}}, k, m$

$\Rightarrow$ higher order elements meaningful for sufficiently smooth solutions.

Loss of smoothness due to

- non-smooth boundaries
- non-smooth coefficients

$\Rightarrow$ For good practical (preasymptotic) convergence:

$|u|_{H^m}$ not too large

high wave numbers?

Practical aspects of FEM

Solution of

$(*')$ Find $u_N \in V_N$ s.t. $a(u_N, v_N) = f(v_N) \quad \forall v_N \in V_N$

Require basis $b_1, \dots, b_N$ of V_N $\Rightarrow$

$$u_N = \sum_{j=1}^{N} \xi_j b_j, \quad \text{Test with } v_N = b_i \text{ for } i = 1, \dots, N$$

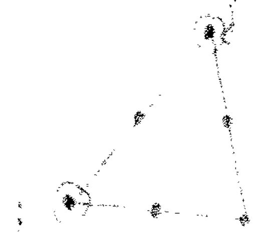
$$(*') \Leftrightarrow A x = g$$

$$(A)_{ij} = a(b_j, b_i)$$

$$(g)_i = f(b_i)$$

local shape functions

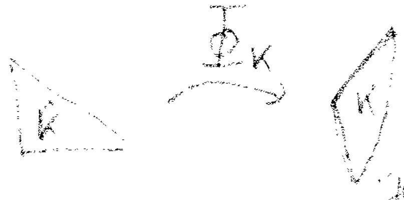
Elementwise assembly



$$(A)_{ij} \leftarrow (A)_{ij} + a(b_j^K, b_i^K)$$

Evaluation of $a(b^K, d^K)$ requires evaluation of integrals,

for example



$$\Phi_K$$

Jacobimatrix

$$\int_K C \nabla_x b^K \cdot \nabla_x d^K \, dx$$

$$= \int_{\hat K} C(\Phi_K \hat x) \cdot \nabla_x b^K(\Phi_K \hat x) \cdot \nabla_x d^K(\Phi_K \hat x) \, |\det J_K(\hat x)| \, d\hat x$$

$$= \int_{\hat K} J_K(\hat x)^{-1} C(\Phi_K \hat x) J_K(\hat x)^{-T} \nabla_{\hat x} \hat b(\Phi_K \hat x) \cdot \nabla_{\hat x} d^K(\Phi_K \hat x) \dots$$

$(\square)$

Δ local op $\Rightarrow$ A zero for FEs with disjoint support ☺ #nnz ~ N

DtN nonlocal op $\Rightarrow$ A dense ☹

Evaluation of integral (□):

- exact if possible
- numerical quadrature

$\left(\|u - u_N\|_{H^1} = O(h_{\mathcal{M}}^p) \right.$

to retain good convergence: Quadrature of order $2p-1$)

Example: 1D - model problem (Section 4.2.1 Ihlenburg)

$$-u'' - k^2 u = f \quad \text{on } \Omega = (0,1)$$

$$u(0) = 0$$

$$u'(1) - iku(1) = 0$$

$$\Rightarrow a(u,v) = \int_0^1 u' \bar{v}' \, dx - k^2 \int_0^1 u \bar{v} \, dx - iK u(1) \bar{v}(1)$$

$$(f,v) = \int_0^1 f \bar{v} \, dx$$

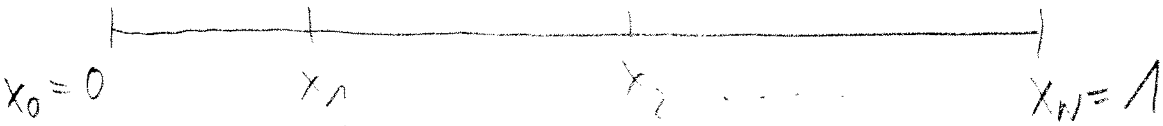
Space $H^1_{(0}(0,1) = V$

Find $u \in V$ s.t. $a(u,v) = f(v) \quad \forall v \in V$.

Exact solution $u(x) = \int_0^1 G(x,s) f(s) \, ds$

with $G(x,s) = \frac{1}{k} \begin{cases} \sin(kx)\, e^{iks} & 0 \le x \le s \\ \sin(ks)\, e^{ikx} & s \le x \le 1 \end{cases}$

Mesh



For simplicity uniform mesh with width h.

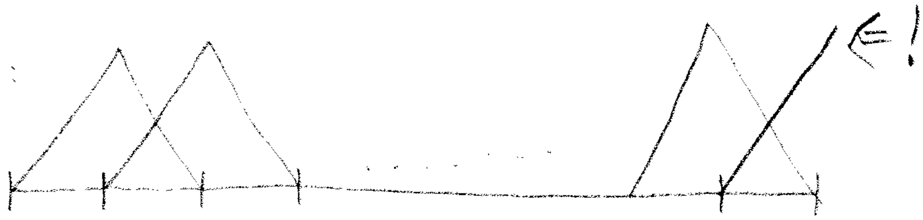
<u>Stability estimates</u>

β = inf-sup constant for model problem

satisfies $\frac{C_1}{k} \le \beta \le \frac{C_2}{k}$ for constants C_1, C_2.

This also holds for discrete inf-sup (see Ihlenburg) if piecewise linear finite elements S_1^{10} are used

S_1^{10}:



Iu = piecewise linear interpolation of u. If $u \in H^2 \rightsquigarrow$

$$\|u - Iu\|_{L^2} \le \left(\frac{h}{\pi}\right)^2 |u|_{H^2}$$

$$|u - Iu|_{H^1} \le \left(\frac{h}{\pi}\right) |u|_{H^2}$$

$$\|u - Iu\|_{L^2} \le \left(\frac{h}{\pi}\right) |u - Iu|_{H^1}$$

u combination of $\sin(kx)$ and $\cos(kx)$ $\rightsquigarrow$

$$\frac{|u|_2}{\|u\|} \le C_1 k^2 \qquad \frac{|u|_2}{|u|_1} \le C_2 k$$

$$\Rightarrow \quad \frac{\|u - Iu\|_{L^2}}{\|u\|_{L^2}} \leq C_1 h^2 K^2 \qquad (*)$$

$$\frac{|u - Iu|_{H^1}}{|u|_{H^1}} \leq C_2 h K$$

$\Rightarrow$ Rule of thumb:

as K grows, h needs to decrease

$$h \sim 1/K$$

to keep error at the same level.

Unfortunately, this is not enough: inf-sup-constant $\sim 1/K$ hits back $\rightsquigarrow$ best approx. error remains constant but Galerkin discretization error grows if $h \sim 1/K$ (see Figure 4.17 in Ihlenburg).

<u>Subtle issue</u>: not important if $h \to 0$ but very important in pre asymptotic phase $hK \gg 0$.

$\rightsquigarrow$ new rule of thumb: $K^3 h^2 \sim 1$ (OK for L^2, $(*)$)