Linear forms, sesquilinear forms and linear operators

I) Linear forms on linear space V and dual space

- $f : V \to \mathbb{C}$ <u>linear form</u> if $f(\alpha(u+v)) = \alpha f(u) + \alpha f(v)$
  $\forall \alpha \in \mathbb{C},\ u \in V,\ v \in V$
- Norm of f induced by norm $\|\cdot\|_V$ on V:
  $$\|f\|_{V'} = \sup_{\substack{v \in V \\ v \neq 0}} \frac{|f(v)|}{\|v\|_V} \qquad (*)$$

(In $\mathbb{C}^n$: $f : x \to y^H x$, norm inducted by Euclidean norm: $\|f\| = \|y\|_2$.)

- V' = linear space of all <u>bounded</u> linear forms
  $=:$ dual space of V
- Unfortunately, the form $u \mapsto \int_\Omega f \cdot \bar{u}\, dx$ (which we will need) is <u>not</u> linear.

  $\Rightarrow f : V \to \mathbb{C}$ is <u>antilinear</u> if $f(\alpha(u+v)) = \bar{\alpha} f(u) + \bar{\alpha} f(v)$

  V^* = linear space of all bounded antilinear forms
  (note that $V' \Leftrightarrow V^*$ by conjugation.)
  $\|f\|_{V^*}$ defined as in $(*)$.
- If V is Hilbertspace with scalarproduct $(\cdot,\cdot)_V$:
  $u \in V \Rightarrow f_u(v) = (u, v)_V$ antilinear form
  Vice versa:
  For every $f \in V^*$ there exists unique $u_f \in V$ s.t.
  $$f(v) = (u_f, v)_V$$
  (Riesz representation theorem)
- Sobolev spaces: $H^{-m} := (H^m)^*$
  Gelfand triple: $H^m \subset L^2 \equiv (L^2)^* \subset H^{-m}$

II) Sesquilinear forms on $V_1 \times V_2$

- $a : V_1 \times V_2 \to \mathbb{C}$ sesquilinear form on linear spaces V_1, V_2 if

$$a(\alpha(u_1+u_2), v) = \alpha\, a(u_1, v) + \alpha\, a(u_2, v) \quad \text{(linear in } 1^{st} \text{ arg.)}$$
$$a(u, \beta(v_1+v_2)) = \bar{\beta}\, a(u, v_1) + \bar{\beta}\, a(u, v_2) \quad \text{(antilinear in } 2^{nd} \text{ arg.)}$$

- Norm of a induced by norms $\|\cdot\|_{V_1}, \|\cdot\|_{V_2}$ on V_1, V_2:

$$\|a\|_{V_1 \times V_2} := \sup_{\substack{u \in V_1 \\ u \neq 0}} \sup_{\substack{v \in V_2 \\ v \neq 0}} \frac{|a(u,v)|}{\|u\|_{V_1} \|v\|_{V_2}}$$

a is called bounded if $\|a\|_{V_1 \times V_2} < \infty$

Example: • $\mathbb{C}^n$: $a(u,v) = v^H A u$, $A \in \mathbb{C}^{n \times n}$

• $H^1(0,1)$: $a(u,v) = \int_0^1 u' v' \, dx$

I) + II) $\Rightarrow$ general linear variational problem:

$$\text{find } \underbrace{u \in V_1}_{\text{trial space}} \text{ such that } \underbrace{a(u,v)}_{\text{bilinear form}} = \underbrace{f}_{\text{antilinear form} \in V_2^*}(v) \quad \forall v \in \underbrace{V_2}_{\text{test space}} \qquad \text{(LVP)}$$

III) Linear operators

- $T : V \to W$ linear operator if $T(\alpha(v_1+v_2)) = \alpha T(v_1) + \alpha T(v_2)$
- norm of T induced by norms on V, W:

$$\|T\|_{V \to W} := \sup_{\substack{v \in V \\ v \neq 0}} \frac{\|Tv\|_W}{\|v\|_V}$$

- $\mathcal{L}(V, W)$ = linear space of all bounded linear operators.

II) + III) $\Rightarrow$ Sesquilinear forms and linear operators

Any sesquilinear form can be turned into linear operator:

$a(u,v)$ sesquilinear form on $V_1 \times V_2$

$\Rightarrow a(u,\cdot)$ antilinear form (V_2^*)

$\Rightarrow A: u \longmapsto a(u,\cdot)$

linear operator $A: V_1 \longrightarrow V_2^*$

$$(LVP) \Rightarrow \quad Au = f$$

$$\|A\|_{V_1 \to V_2^*} = \|a\|_{V_1 \times V_2}$$

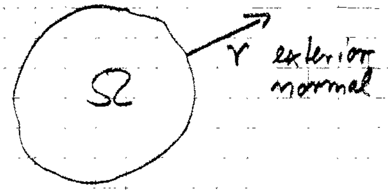
Variational Formulation of Helmholtz Problems

Helmholtz with mixed boundary conditions

$$\boxed{\begin{array}{ll} -\Delta u - k^2 u = 0 & \text{in } \Omega \\ \partial_\nu u + \beta u = g & \text{on } \partial\Omega \end{array}}$$

$\partial_\nu u = \nabla u \cdot \nu$

β: complex constant



bounded domain Ω

For Neumann and mixed boundary conditions: test and trial space H^1.

Find $u \in H^1(\Omega)$ (trial space) such that for all $v \in H^1(\Omega)$ (test space):

$$-\int_\Omega (\underbrace{\Delta u}_{\text{integration by parts}} + \underbrace{k^2 u}_{\text{don't touch}})\, \bar{v}\, dx = 0$$

$$\Downarrow$$

$$0 = -\int_{\partial\Omega} \partial_\nu u \cdot \bar{v}\, dS + \int_\Omega \nabla u \cdot \nabla \bar{v}\, dx - \int_\Omega k^2 u \bar{v}\, dx$$

boundary cond. ↳

$$= \underbrace{-\int_{\partial\Omega} g \cdot \bar{v}\, dS}_{= -f(v)} + \underbrace{\beta \int_{\partial\Omega} u \bar{v}\, dS + \int_\Omega \nabla u \cdot \nabla \bar{v}\, dx - \int_\Omega k^2 u \bar{v}\, dx}_{= a(u,v)}$$

$\Rightarrow$ Variational formulation

Find $u \in H^1(\Omega)$ s.t. $a(u,v) = f(v) \quad \forall v \in H^1(\Omega)$

(About the r.h.s.: $v \in H^1 \Rightarrow$ trace of $v \in H^{1/2} \Rightarrow g$ must be in $H^{-1/2}$ for $\int_{\partial\Omega} g \cdot v\, dS$ to remain bounded

$\Rightarrow L^2 \subset H^{-1/2}$, $g \in L^2$ is fine.)

Well-Posedness of Variational Problems

Part I Positive definite form

Motivation If matrix $A \in \mathbb{C}^{n\times n}$ is positive definite ($u^H A u > 0 \quad \forall u \in \mathbb{C}^n, u \neq 0$) $\Rightarrow$ A is invertible.

sesquilinear form $a : V \times V \to \mathbb{C}$ called V-elliptic if

$$|a(u,u)| \geq \alpha \|u\|_V^2 \qquad \forall u \in V$$

for some $\alpha > 0$.

Theorem (Lax-Milgram)

If $a: V \times V \to \mathbb{C}$ bounded sesquilinear form and V-elliptic then

$$a(u,v) = f(v) \qquad (*)$$

has unique solution u for every $f \in V^*$.

Remark: If $M = \|a\|_{V\times V}$ and α ellipticity constant $\overset{(*)}{\Rightarrow}$

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V^*}$$

$$\Rightarrow \alpha \|u\|_V^2 \leq |a(u,u)| \leq M \|u\|_V^2$$

$\Rightarrow$ energy norm $\sqrt{|a(u,u)|}$ equivalent to $\|u\|_V$.

Example:

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad \Rightarrow \quad a(u,v) = \int_\Omega \nabla u \, \nabla \bar{v} \, dx$$

$V = H_0^1(\Omega)$, H^1-norm

$$\Rightarrow |a(u,u)| = \|\nabla u\|_{L^2}^2$$

For V-ellipticity we also need Poincaré inequality (! $u \in H_0^1$ required): $\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}$

$$\Rightarrow a(u,u) \geq \frac{1}{1+C^2} \|u\|_{H^1}^2$$

Part II. The inf-sup conditions

Helmholtz is indefinite and does not fit in framework of V-ellipticity.

Motivation: Smallest singular value of matrix A satisfies

$$\sigma_{min} = \inf_{\substack{u \in \mathbb{C}^n \\ u \neq 0}} \frac{\|Au\|_2}{\|u\|_2}$$

$$= \inf_{\substack{u \in \mathbb{C}^n \\ u \neq 0}} \sup_{\substack{v \in \mathbb{C}^n \\ v \neq 0}} \frac{|v^* A u|}{\|u\|_2 \, \|v\|_2}$$

$$\sigma_{min} > 0 \quad \Leftrightarrow \quad A \text{ invertible}$$

weakest solvability requirement

Theorem (Babuška)

If $a: V_1 \times V_2 \to \mathbb{C}$ bounded sesquilinear form and

- (inf-sup condition 1) $$\beta = \inf_{\substack{u \in V_1 \\ u \neq 0}} \sup_{\substack{v \in V_2 \\ v \neq 0}} \frac{|a(u,v)|}{\|u\|_{V_1} \, \|v\|_{V_2}} > 0$$
- (inf-sup condition 2 ~ related to surjectivity) $$\sup_{\substack{u \in V_1 \\ u \neq 0}} |a(u,v)| > 0 \qquad \forall v \in V_2,\ v \neq 0.$$

Then $a(u,v) = f(v) \quad \forall v \in V_2$ has unique solution for all $f \in V_2^*$. Moreover,

$$\|u\|_{V_1} \leq \frac{1}{\beta} \|f\|_{V_2^*}.$$

The opposite of Babuška's theorem is also true.

Coercivity

$V = H^1(\Omega)$

- Sesquilinear form a $\underline{V\text{-coercive}}$ if it satisfies

$$|a(u,u) + C\|u\|_{L^2}^2| \geq \alpha \|u\|_{H^1}^2 \quad \forall u \in V$$

for some constants $\alpha, C > 0$.

- $\hat{=}$ V-ellipticity of $\tilde{a}(u,v) = a(u,v) + C(u,v)_{L^2}$

Observation: a compact perturbation (embedding $H^1 \subset L^2$ compact) of V-elliptic operator

$\Rightarrow$ Fredholm alternative

either	or
(LVP) has a solution for all $f \in V^*$	$\exists$ nontrivial solution of homog. problem ($f = 0$)

(Remark: trivial for finite-dimensional spaces)

$\Rightarrow$ Only need to show uniqueness of homogeneous problem!

Example:

$$\boxed{\begin{aligned} -u'' - k^2 u &= f \quad \text{on } \Omega = (0,1) \\ u(0) &= 0 \\ u'(1) - i \underset{\in \mathbb{R}^+}{\alpha}\, u(1) &= 0 \end{aligned}} \qquad (*)$$

$\Downarrow$ $H^1_{(0}(0,1) = \{ u \in H^1(0,1) : u|_0 = 0 \}$

$$a(u,v) = \int_0^1 (u' \bar{v}' - k^2 u \bar{v})\, dx - i \alpha u(1) \bar{v}(1).$$

Gårding's inequality is satisfied:

$$a(u,u) = \|u\|_{H^1}^2 - k^2 \|u\|_{L^2}^2, \quad \text{set } C = k^2$$

Homogeneous problem $a(u,v)=0$

$$\Rightarrow a(u,u)=0 \Rightarrow \int_0^1 (|u'|^2 - k^2|u|^2)\,dx = i\alpha |u(1)|^2$$

real

purely imaginary

$\alpha > 0$

$\Rightarrow u(1)=0$

boundary condition $\Rightarrow$ $u'(1)=0 \Rightarrow$ $\begin{array}{l} -u''-k^2u=f \\ u(1)=0 \\ u'(1)=0 \end{array}$ initial value problem with unique solution $u \equiv 0$

$\Uparrow$

! requires $u\in H^2$ to get trace of u' !
(ok if $f \in L^2$)

$$a(u,v)=f(v) \quad \forall v \in H_{(0}(0,1)$$

has unique solution $u \in H_{(0}(0,1)$.