Wave equations and Helmholtz equations in physics

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Maxwell's equations

$$
\begin{aligned}
-\frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} &= \mathbf{J}, && \text{Ampère's circulation law} \\
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0, && \text{Farady's law of induction} \\
\nabla \cdot \mathbf{D} &= \rho, && \text{Gauß's law} \\
\nabla \cdot \mathbf{B} &= 0, && \text{Gauß's law for magnetism,}
\end{aligned} \tag{1}
$$

Charge conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

More information in Ihlenburg, Chapter 1

Linear and isotropic constitutive relations

$$\begin{aligned}
\mathbf{D}(x,t) &= \epsilon_{\mathrm{s}} * \mathbf{E} = \int_{-\infty}^{t} \epsilon_{\mathrm{s}}(x,t-s)\mathbf{E}(x,s)\,\mathrm{d}s, \\
\mathbf{B}(x,t) &= \mu * \mathbf{H} = \int_{-\infty}^{t} \mu(x,t-s)\mathbf{H}(x,s)\,\mathrm{d}s, \\
\mathbf{J}(x,t) &= \sigma * \mathbf{E} + \mathbf{J}_{\mathrm{e}} = \int_{-\infty}^{t} \sigma(x,t-s)\mathbf{E}(x,s)\,\mathrm{d}s + \mathbf{J}_{\mathrm{e}}(x,t).
\end{aligned} \tag{2}$$

A convolution in time express a non-instantaneous response of the material to an applied field.

The Fourier transform

Assume f, g smooth in t and has compact support ($f(t) = 0$ when $|t| > t_0$). Define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\omega} f(t)\,\mathrm{dt}. \tag{3}$$

Excercise 1: Derive $\hat{f}'(t)$. Integration by parts gives

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\omega} f'(t)\,\mathrm{dt} = -\int_{-\infty}^{\infty} \mathrm{i}\omega \mathrm{e}^{\mathrm{i}t\omega} f(t)\,\mathrm{dt} = -\mathrm{i}\omega\hat{f}(\omega) \tag{4}$$

The Fourier transform

We also need the Fourier transform of $f * g$:

$$\begin{aligned}\widehat{(f * g)}(\omega) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\omega} \int_{-\infty}^{\infty} f(t-s)g(s)\,\mathrm{dsdt} \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s\omega} g(s) \left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(t-s)\omega} f(t-s)\,\mathrm{dt} \right) \mathrm{ds} \qquad (5)\end{aligned}$$

$$= \hat{f}(\omega)\hat{g}(\omega)$$

More general

The Schwartz space:

- $\mathcal{S}(\mathbb{R})$ denote the set of $C^\infty -$ functions that are rapidly decreasing at ∞
- Example: $\mathrm{e}^{-|x|^2}$

The dual space:

- The dual space $\mathcal{S}'(\mathbb{R})$ is called **tempered distributions**
- Example: δ, polynomials, ...

More general

The Fourier transform $\hat{f}$ of $f \in \mathcal{S}'(\mathbb{R})$ is defined by

$$\hat{f}(\phi) = f(\hat{\phi}), \quad \phi \in \mathcal{S}(\mathbb{R}). \tag{6}$$

Example: Assume $\int_{-\infty}^{\infty} |f|, \mathrm{d}t < \infty$. Then

$$\begin{aligned} \hat{f}(\phi) &= \int_{-\infty}^{\infty} f(\omega)\hat{\phi}(\omega), \mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} f(\omega) \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\omega}\phi(t)\mathrm{d}t\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\omega)\mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\omega \right) \phi(t)\mathrm{d}t \end{aligned} \tag{7}$$

More general (for mathematicians)

We have

- $\hat{f}'(\omega) = -\mathrm{i}\omega\hat{f}(\omega)$ holds for $f \in \mathcal{S}'(\mathbb{R})$
- $\widehat{(f * g)}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$ holds when $f, g \in \mathcal{S}'(\mathbb{R})$ and $g = 0$ for $t < t_0$.

Resonances in open systems

- Resonances corresponds to a complex frequency
- The formulas above also holds for a complex frequency $\omega = \omega' + \mathrm{i}\omega''$ for which $\mathrm{e}^{-\omega'' t} f \in \mathcal{S}'(\mathbb{R})$.

Exercise 1.1

Take the Fourier transform of Maxwell's equations and the charge conservation:

$$
\begin{aligned}
\mathrm{i}\omega\hat{\epsilon}_s\hat{\mathbf{E}} + \nabla \times \hat{\mathbf{H}} &= \hat{\mathbf{J}}, \quad \hat{\mathbf{J}} = \hat{\sigma}\hat{\mathbf{E}} + \hat{\mathbf{J}}_{\mathrm{e}} \\
-\mathrm{i}\omega\hat{\mu}\hat{\mathbf{H}} + \nabla \times \hat{\mathbf{E}} &= 0, \\
\nabla \cdot (\hat{\epsilon}_s\hat{\mathbf{E}}) &= \hat{\rho}, \\
\nabla \cdot (\hat{\mu}\hat{\mathbf{H}}) &= 0, \\
-\mathrm{i}\omega\hat{\rho} + \nabla \cdot \hat{\mathbf{J}} &= 0 \quad \Leftrightarrow \hat{\rho} = -\frac{\mathrm{i}}{\omega}\nabla \cdot \hat{\mathbf{J}}
\end{aligned}
\tag{8}
$$

$\Rightarrow$

Exercise 1.1

$$
\begin{aligned}
\mathrm{i}\omega \left(\hat{\epsilon}_{\mathrm{s}}(\tilde{\omega}) + \mathrm{i}\frac{\sigma(\tilde{\omega})}{\tilde{\omega}} \right) \hat{\mathbf{E}} + \nabla \times \hat{\mathbf{H}} = \hat{\mathbf{J}}_{\mathrm{e}} \\
-\mathrm{i}\omega\hat{\mu}\hat{\mathbf{H}} + \nabla \times \hat{\mathbf{E}} = 0, \\
\nabla \cdot \left(\left(\hat{\epsilon}_{\mathrm{s}}(\tilde{\omega}) + \mathrm{i}\frac{\sigma(\tilde{\omega})}{\tilde{\omega}} \right) \hat{\mathbf{E}} \right) = -\frac{\mathrm{i}}{\omega} \nabla \cdot \hat{\mathbf{J}}_{\mathrm{e}}, \\
\nabla \cdot (\hat{\mu}\hat{\mathbf{H}}) = 0.
\end{aligned}
\tag{9}
$$

The complex permittivity is given by

$$
\hat{\epsilon} = \hat{\epsilon}_{\mathrm{s}}(\tilde{\omega}) + \mathrm{i}\frac{\sigma(\tilde{\omega})}{\tilde{\omega}} \tag{10}
$$

Exercise 1.2

Let $\hat{\epsilon} = \epsilon_0$ (the electric constant) and $\hat{\mu} = \mu_0$ (the magnetic constant):

$$\begin{aligned} \mathrm{i}\omega\epsilon_0 \hat{\mathbf{E}} + \nabla \times \hat{\mathbf{H}} &= \hat{\mathbf{J}}_{\mathrm{e}} \\ -\mathrm{i}\omega\mu_0 \hat{\mathbf{H}} + \nabla \times \hat{\mathbf{E}} &= 0, \\ \nabla \cdot \hat{\mathbf{E}} &= -\frac{\mathrm{i}}{\epsilon_0 \omega} \nabla \cdot \hat{\mathbf{J}}_{\mathrm{e}}, \\ \nabla \cdot \hat{\mathbf{H}} &= 0. \end{aligned} \tag{11}$$

$\nabla\times$ (Eq 2) $\Rightarrow$

$$-\mathrm{i}\omega\mu_0 \underbrace{\nabla \times \hat{\mathbf{H}}}_{\hat{\mathbf{J}}_{\mathrm{e}} - \mathrm{i}\omega\epsilon_0 \hat{\mathbf{E}}} + \underbrace{\nabla \times (\nabla \times \hat{\mathbf{E}})}_{-\Delta \hat{\mathbf{E}} + \nabla(\nabla \cdot \hat{\mathbf{E}})} = 0 \tag{12}$$

Exercise 1.2

$\nabla \cdot \hat{\mathbf{E}} = -\frac{\mathrm{i}}{\epsilon_0 \omega} \nabla \cdot \hat{\mathbf{J}}_\mathrm{e} \Rightarrow$

$$-\Delta \hat{\mathbf{E}} - \frac{\mathrm{i}}{\epsilon_0 \omega} \nabla(\nabla \cdot \hat{\mathbf{J}}_\mathrm{e}) - \mathrm{i}\omega\mu_0 \hat{\mathbf{J}}_\mathrm{e} - \omega^2 \epsilon_0 \mu_0 \hat{\mathbf{E}} = 0 \tag{13}$$

$\Leftrightarrow$

$$-\Delta \hat{\mathbf{E}} - \omega^2 \epsilon_0 \mu_0 \hat{\mathbf{E}} = \mathbf{F}, \quad \mathbf{F} = \frac{\mathrm{i}}{\epsilon_0 \omega} \nabla(\nabla \cdot \hat{\mathbf{J}}_\mathrm{e}) + \mathrm{i}\omega\mu_0 \hat{\mathbf{J}}_\mathrm{e} \tag{14}$$

Let $c = (\epsilon_0 \mu_0)^{-1/2}$ denote the speed of light. The vector Helmholtz equation is then

$$-\Delta \hat{\mathbf{E}} - k^2 \hat{\mathbf{E}} = \mathbf{F}, \quad k = \omega / c \tag{15}$$

Exercise 1.2

$\nabla\times$ (Eq 1) $\Rightarrow$

$$i\omega\epsilon_0 \underbrace{\nabla \times \hat{\mathbf{E}}}_{i\omega\mu_0\hat{\mathbf{H}}} + \underbrace{\nabla \times (\nabla \times \hat{\mathbf{H}})}_{-\Delta\hat{\mathbf{H}}} = \nabla \times \hat{\mathbf{J}}_e \tag{16}$$

Exercise 1.3, TM-polarized waves

Transverse magnetic (TM) polarized wave: $(0, 0, E_3, H_1, H_2, 0)$:
From Eq. 2 follows

$$\hat{\mathbf{H}} = \frac{1}{\mathrm{i}\omega\hat{\mu}} \nabla \times \hat{\mathbf{E}}, \quad \nabla \times \hat{\mathbf{E}} = (\partial_2 E_3, -\partial_1 E_3, 0) \tag{17}$$

Eq. 1 gives

$$\mathrm{i}\omega\hat{\epsilon}\hat{\mathbf{E}} + \frac{1}{\mathrm{i}\omega} \underbrace{\nabla \times \left[\frac{1}{\hat{\mu}}(\nabla \times \hat{\mathbf{E}})\right]}_{(0,0,-\nabla\cdot\left(\frac{1}{\hat{\mu}}\nabla E_3\right))} = \left(\hat{\mathbf{J}}_{\mathrm{e}}\right)_3 \tag{18}$$

$\Rightarrow$

$$-\nabla \cdot \left(\frac{1}{\hat{\mu}} \nabla E_3\right) - \omega^2 \hat{\epsilon} E_3 = \mathrm{i}\omega \left(\hat{\mathbf{J}}_{\mathrm{e}}\right)_3 \tag{19}$$

Exercise 1.3, TE-polarized waves

Transverse magnetic (TM) polarized wave: $(E_1, E_2, 0, 0, 0, H_3)$:
From Eq. 1 follows

$$\hat{\mathbf{E}} = -\frac{1}{\mathrm{i}\omega\hat{\epsilon}}\nabla \times \hat{\mathbf{H}} + \frac{1}{\mathrm{i}\omega\hat{\epsilon}}\hat{\mathbf{J}}_{\mathrm{e}} \tag{20}$$

Eq. 2 gives

$$-\mathrm{i}\omega\hat{\mu}\hat{\mathbf{E}} - \frac{1}{\mathrm{i}\omega}\nabla \times \left[\frac{1}{\hat{\epsilon}}(\nabla \times \hat{\mathbf{H}})\right] = -\frac{1}{\mathrm{i}\omega}\nabla \times \left(\frac{1}{\hat{\epsilon}}\hat{\mathbf{J}}_{\mathrm{e}}\right) \tag{21}$$

$\Rightarrow$

$$-\nabla \cdot \left(\frac{1}{\hat{\epsilon}}\nabla H_3\right) - \omega^2\hat{\mu}H_3 = \nabla \times \left(\frac{1}{\hat{\epsilon}}\hat{\mathbf{J}}_{\mathrm{e}}\right)_3 \tag{22}$$

Linear Acoustics

$$-\Delta u - k^2 u = 0. \tag{23}$$

- u - acoustic pressure or velocity potential
- $u = p$ - on a surface is called *acoustically soft*
- $\nabla u \cdot \mathbf{n} = ikv$ - on a surface is called *acoustically hard*

More information in Ihlenburg, Chapter 1

Quantum mechanics

The time-independent Schrödinger equation (one particle, stationary states):

$$\frac{\hbar}{2m}\Delta\Psi + (E - V)\Psi = 0 \qquad (24)$$

- $\hbar = h/2\pi$ - h is the Planck constant
- m - mass of the particle
- Ψ - the probability density
- V - potential energy
- E - energy of the particle

In general

- The wave equation (or more general hyperbolic partial differential equations) describe a large variety of phenomena in physics
- Helmholtz equation is the time-independent form of a wave equation (also used for parabolic problems, such as the heat equation)