END of Rellich's lemma

Previously,

Spherical Bessel functions:

$$j_n(\xi) := \sum_{p=0}^{\infty} \frac{(-1)^p \xi^{n+2p}}{2^p p!\, 1\cdot 3 \cdots (2n+2p+1)}$$

$$y_n(\xi) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p \xi^{2p-n-1}}{2^p p!\, (-2n+1) \cdots (-2n+2p+1)}$$

$$h_n^{(1)} := j_n + i y_n \quad , \quad h_n^{(2)} := j_n - i y_n$$

- Wronskian $\quad W(j_n, y_n) := j_n(\xi)\, y_n'(\xi) - y_n(\xi)\, j_n'(\xi)$

$$= \frac{1}{\xi^2}$$

We stated,

Theorem. Let Y_n be a spherical harmonic of order n. Then

$$u_n(x) = j_n(k|x|)\, Y_n(\hat{x})$$

is an entire solution to the Helmholtz equation and

$$v_n(x) = h_n^{(1)}(k|x|)\, Y_n(\hat{x})$$

is a radiating solution to Helmholtz in $\mathbb{R}^3 \setminus \{0\}$.

Entire: solution of homogeneous Helmholtz over $\mathbb{R}^3$

Radiating: " " " over part of $\mathbb{R}^3$ satisfying the radiation condition

Note!

$\Rightarrow$ entire + radiating condition $\Rightarrow u \equiv 0$ everywhere.

Proof: We can write $j_n(kr) = k^n r^n w_n(r^2)$ with $w_n : \mathbb{R} \to \mathbb{R}$ analytic. Since $r^n Y_n(\hat{x})$ is a homogeneous polynomial in x_1, x_2, x_3, the product $j_n(kr) Y_n(\hat{x})$ is regular at $x = 0$, i.e. u satisfies Helmholtz at the origin (and everywhere by construction). Radiation condition follows from the asymptotic behavior from Hankel functions.

Theorem: (Addition theorem)

Let Y_n^m, $m = -n, \dots, n$, $n = 0, 1 \dots$ be a set of orthonormal spherical harmonics. Then for $|x| > |y|$ we have

$$(*) \qquad \frac{e^{ik|x-y|}}{4\pi |x-y|} = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k|x|)\, Y_n^m(\hat{x})\, j_n(k|y|)\, \overline{Y_n^m(\hat{y})}$$

$\hat{x} = x/|x|$, $\hat{y} = y/|y|$.

The series and term by term derivatives with respect to $|x|$ and $|y|$ are absolutely and uniformly convergent on compact subsets of $|x| > |y|$

Proof:

Green's theorem applied to $u_n^m(z) = j_n(k|z|)\, Y_n^m(\hat{z})$ and $G(x,y)$

$$\int_{|z|=r} \left(u_n^m(z) \frac{\partial G(x,z)}{\partial \nu(z)} - \frac{\partial u_n^m}{\partial \nu(z)} G(x,z) \right) ds(z) = 0 \qquad |x| > r$$

Representation formula applied to $v_n^m = h_n^{(1)}(k|z|)\, Y_n^m(\hat{z})$

$$\int_{|z|=r} \left\{ v_n^m(z) \frac{\partial G(x,z)}{\partial \nu(z)} - \frac{\partial v_n^m}{\partial \nu(z)} G(x,z) \right\} d\sigma(z) = v_n^m(z)$$

$|x| > r$

On $|z| = r$ we have

$$u_n^m(z) = j_n(kr)\, Y_n^m(\hat{z}), \quad \frac{\partial u_n^m}{\partial \nu} = k\, j_n'(kr)\, Y_n^m(\hat{z})$$
$$v_n^m(z) = h_n^{(1)}(kr)\, Y_n^m(\hat{z}), \quad \frac{\partial v_n^m}{\partial \nu} = k\, h_n^{(1)'}(kr)\, Y_n^m(\hat{z})$$

Using the Wronskian [Ex]

$$\frac{1}{ikr^2} \int_{|z|=r} Y_n^m(\hat{z})\, G(x,z)\, d\sigma(z) = j_n(kr)\, h_n^{(1)}(k|x|)\, Y_n^m(\hat{x})$$

$|x| > r$

By integrating in the unit sphere

$$\int_{S^2} Y_n^m(\hat{z})\, G(x, r\hat{z})\, ds(\hat{z}) = ik\, j_n(kr)\, h_n^{(1)}(k|x|)\, Y_n^m(\hat{x})$$

$|x| > r$

By using Y_n^m as a basis, we can write

$$G(x,y) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\int_{S^2} Y_n^m(\hat{z})\, G(x, r\hat{z})\, d\sigma(\hat{z}) \right) \overline{Y_n^m}(\hat{y})$$

the series (*) is valid for fixed x, $|x| > r$ in y in L^2-sense on the sphere $|y| = r$ for arbitrary r.

we can show the estimate [Ex]

$$\sum_{m=-n}^{n} \left| h_n^{(1)}(k|x|)\, Y_n^m(\hat{x})\, j_n(k|y|)\, \overline{Y_n^m}(\hat{y}) \right| \le$$

$\leq \frac{2n+1}{4\pi} \left| h_n^{(1)}(k|x|)\, j_n(k|y|) \right| = \mathcal{O}\left(\frac{|y|^n}{|x|^n} \right) \quad n \to \infty$ (4)

uniformly on compact sets $|x| > |y|$.
This implies absolute and uniform convergence of series (a).
The case of derivatives is left as an exercise. ∎

<u>Two Formulas</u> [Ex]

<u>Funk - Hecke</u>:
$$\int_{S^2} e^{-ikr\hat{x}\cdot\hat{y}}\, Y_n(\hat{y})\, ds = \frac{4\pi}{i^n} j_n(kr)\, Y_n(\hat{x})$$
$\hat{x} \in S^2,\ r > 0$

<u>Jacobi - Anger expansion</u>:
$$e^{ikx\cdot d} = \sum_{n=0}^{\infty} i^n (2n+1)\, j_n(k|x|)\, P_n(\cos\theta) \qquad x \in \mathbb{R}^3$$
where d is a unit vector, θ denotes the angle between x and d and the convergence is uniform on compact sets of $\mathbb{R}^3$.

<u>Back to Rellich's lemma</u>

<u>Lemma</u>: Assume D bounded, $u \in H^2_{loc}(\mathbb{R}^3 \setminus \overline{D})$ be a ~~radia~~ solution to Helmholtz satisfying
$$\lim_{r\to\infty} \int_{|x|=r} |u(x)|^2\, ds = 0$$
then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Proof: For sufficiently large $|x|$, by the orthonormal basis of $L^2(S^2)$, we have a Fourier expansion

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m(|x|)\, Y_n^m(\hat{x})$$

Coefficients are given by

$$a_n^m(r) = \int_{S^2} u(r\hat{x})\, \overline{Y_n^m}(\hat{x})\, d\sigma(\hat{x})$$

and satisfy Parseval's equality

$$\int_{|x|=r} |u(x)|^2 d\sigma = r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a_n^m(r)|^2$$

By Hypothesis, $\lim_{r\to\infty} r^2 |a_n^m(r)|^2 = 0 \quad \forall n,m.$ (**)

On the other hand, since $u \in H^2_{loc}(\mathbb{R}^3 \setminus \bar{D})$ we can differentiate under the integral. Integration by parts yields

$$\frac{d^2 a_n^m}{dr^2} + \frac{2}{r}\frac{d a_n^m}{dr} + \left(k^2 - \frac{n(n+1)}{r^2}\right) a_n^m = 0$$

(Bessel diff. equation)

Solutions are $a_n^m(r) = \alpha_n^m h_n^{(1)}(kr) + \beta_n^m h_n^{(2)}(kr)$.

Substituting in (**) and using the asymptotic behavior for Hankel functions yields $\alpha_n^m = \beta_n^m = 0 \ \forall n,m$.

$\Rightarrow$ $u = 0$ outside a sufficiently large sphere and $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ by analyticity.