Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Departement Informationstechnologie und Elektrotechnik

Analysis and Implementation of Absorbing Boundary Conditions by Matlab

Formatvorlage des Untertitelmasters durch Klicken bearbeiten

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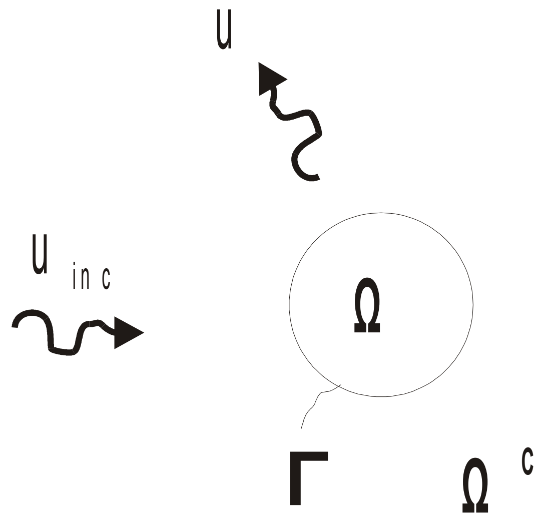
Introduction: unbounded problem

- Scattering of plane wave uinc from bounded object in unbounded medium

$$-\mathrm{div}\,\alpha(\mathbf{x})\nabla u^{tot}(\mathbf{x}) - k^2\beta(\mathbf{x})u^{tot}(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega^c \cup \Omega$$

$$u^{tot} = u^{inc} + u$$



- Not well defined, unless Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left|\frac{\partial u}{\partial r} - iku\right| = 0$$

- Not possible to solve this problem numerically

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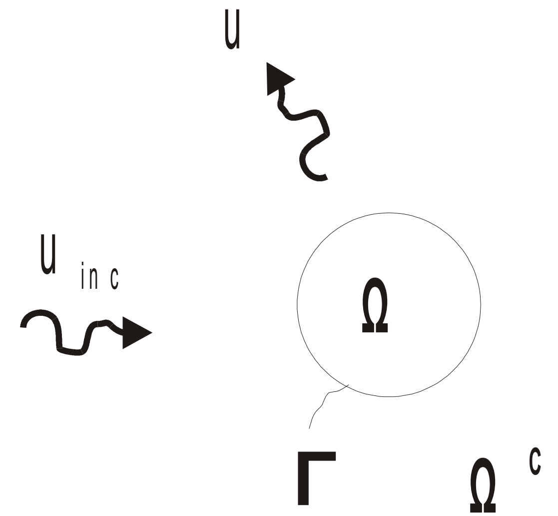

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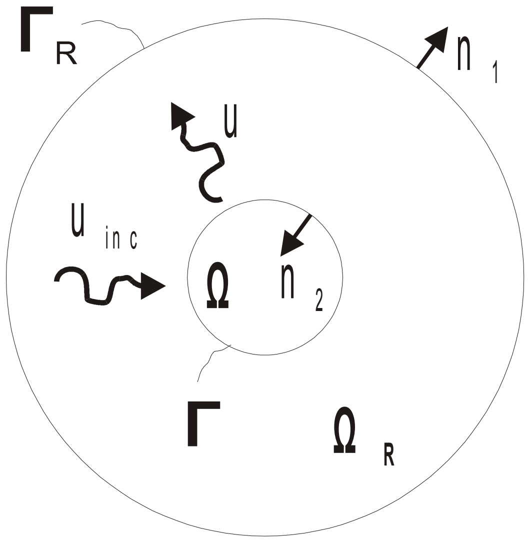
Bound the domain with non-reflecting artificial surface

Introduction: bounded domain

- Scatterer is PEC
- Artificial boundary at r=R



$$-\alpha_0 \Delta u - k^2 \beta_0 u = 0 \quad \mathbf{x} \in \Omega_R$$

$$u = -u^{inc} \quad \text{on} \quad \Gamma$$

$$\frac{\partial u}{\partial r} = Gu \quad \text{on } \Gamma_R$$

Introduction: bounded domain

- G linear differential operator @ r=R, (ΓR)
- Sommerfeld radiation condition is very inaccurate
- Increased accuracy with
  - higher differential orders of G
    - Reflections of higher order modes can be annihilated
  - Bigger radius R of bounded domain
    - Higher order modes decay fast

ABCs in bounded domains

- DtN ABCs based on convergent expansion:

$$u = H_0(kr) \sum_{j=0}^{\infty} \frac{F_j(\theta)}{r^j} + H_1(kr) \sum_{j=0}^{\infty} \frac{G_j(\theta)}{r^j}$$

- BGT ABCs based on asymptotic expansion for kr>>1 [1]:

$$u \sim \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/2)} \sum_{j=0}^{\infty} \frac{f_j(\theta)}{r^j}$$

[1] A. Bayliss, M. Gunzburger, and E. Turkel. ”Boundary conditions for the numerical solution of elliptic equations in exterior regions”. *SIAM Journal of Applied Mathematics*, 42(2):430–450, 1982.

ABCs in bounded domains

- Considered boundary conditions [2]:

- BGT-1

$$G = ik - \frac{1}{2R}$$

- BGT-2

$$G = ik - \frac{1}{2R} + \frac{1}{8R(1-ikR)} + \frac{1}{2R(1-ikR)}\frac{\partial^2}{\partial\theta^2}$$

- DtN-1

$$G = k\frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)}$$

- DtN-2

$$G = k\left(\frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)} + \left(\frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)} - \frac{H_1^{(1)'}(kR)}{H_1^{(1)}(kR)}\right)\frac{\partial^2}{\partial\theta^2}\right)$$

[2] I. Harari and R. Djellouli. "Analytical study of the effect of wave number on the performance of local absorbing boundary conditions for acoustic scattering". *Applied numerical mathematics*, 50:15–47, 2004.

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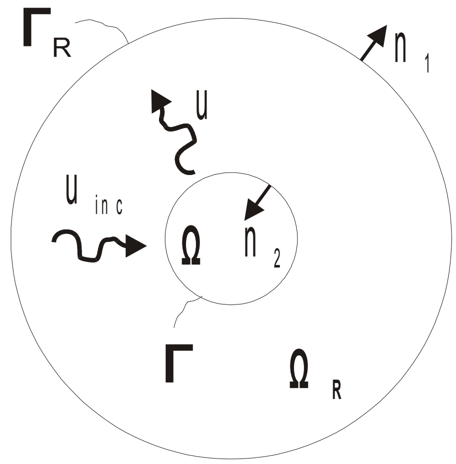


Variational formulation

- Multiply with test-function v є H1 and integrate over ΩR, with boundary conditions:

$$\langle g, v\rangle_{\Gamma} = B(u,v) - \langle Gu, v\rangle_{\Gamma_R}$$

$$0 = \underbrace{-\alpha_0 \int_{\Gamma} \frac{\partial u^{inc}}{\partial r}\bar{v}ds}_{\langle g,v\rangle_{\Gamma}} \underbrace{-(\alpha_0 \int_{\Gamma_R} \left(ik - \frac{1}{2R} + \frac{1}{8R(1-ikR)}\right) u\bar{v}ds + \alpha_0 \int_{\Gamma_R} \frac{1}{2R(1-ikR)}\frac{\partial^2 u}{\partial\theta^2}\bar{v}ds)}_{\langle Gu,v\rangle_{\Gamma_R}} + \underbrace{\alpha_0 \int_{\Omega_R} \nabla\bar{v}\nabla u dx - k^2\beta_0 \int_{\Omega_R} u\bar{v}dx}_{B(u,v)}$$

Uniqueness of solution?

- Uniqueness for coupled problem (see Lecture 12,[3],[4]):

$$Im\langle Gu, u\rangle > 0$$

- DtN-1 was proved (Lecture 12)
- For large kR uniqueness of BGT-1 follows from:
- DtN-2 was shown (previous project)

[3] Frank Ihlenburg. *"Finite Element Analysis of Acoustic Scattering"*. Springer, 1998.
[4] M. J. Grote and J. B. Keller. "On Nonreflecting Boundary Conditions". *Journal of Computational Physics*, 122:231–243, 1995.

Uniqueness: BGT-2

- To show: $Im\langle Gu, u\rangle > 0$

$$Im\langle Gu, u\rangle = \alpha_0 k\left(\left(1 + \frac{1}{8(1+k^2R^2)}\right)\int_{\Gamma_R} |u|^2 ds - \frac{1}{2(1+k^2R^2)}\int_{\Gamma_R} |\frac{\partial^2 u}{\theta^2}|^2 ds\right)$$

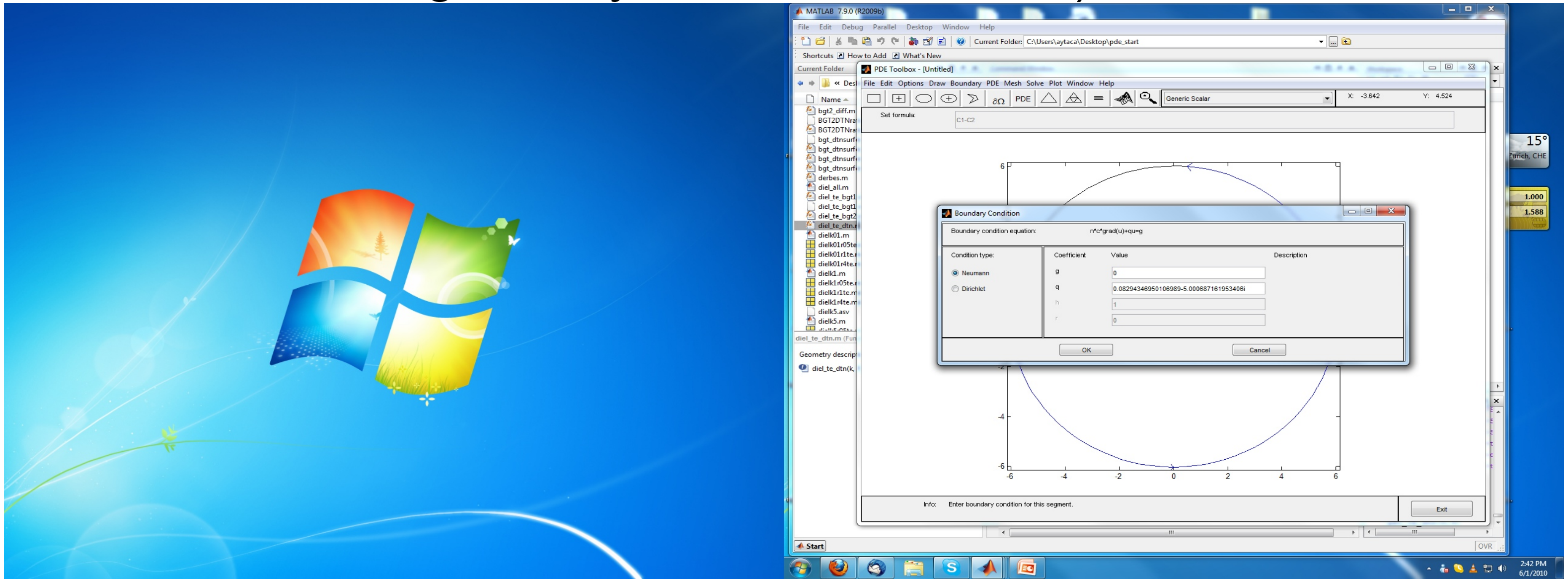
- BGT-2 not unique for all kR!!

- For large kR, we have

$$Im\langle Gu, u\rangle \approx \alpha_0 k \int_{\Gamma_R} |u|^2 ds > 0$$

Implementation in Matlab

- Robin Boundary Conditions availible:
  - BGT-1, DtN-1 already availible
  - Matrix locations already availible for higher order terms of BGT-2 and DtN-2 (boundary mass – boundary stiffness matrix entry locations in the global system are the same)



Higher order terms of BGT-2 and DtN-2

- $\int_{\Gamma_R} u\bar{v}\,dS$ is available in `assemb` routine as (with linear basis functions):

```
if any(q(:)) | any(isnan(q(:)))
  qd=(q.*(ones(N*N,1)*d/3)); % Diagonal values
  qod=0.5*qd; % Off diagonal values
  m=1;
  for l=1:N,
    for k=1:N,
      Q=Q+sparse(ie1+(k-1)*np,ie2+(l-1)*np,qod(m,:),N*np,N*np);
      Q=Q+sparse(ie2+(k-1)*np,ie1+(l-1)*np,qod(m,:),N*np,N*np);
      Q=Q+sparse(ie1+(k-1)*np,ie1+(l-1)*np,qd(m,:),N*np,N*np);
      Q=Q+sparse(ie2+(k-1)*np,ie2+(l-1)*np,qd(m,:),N*np,N*np);
      m=m+1;
    end
  end
end
```

boundary mass matrix

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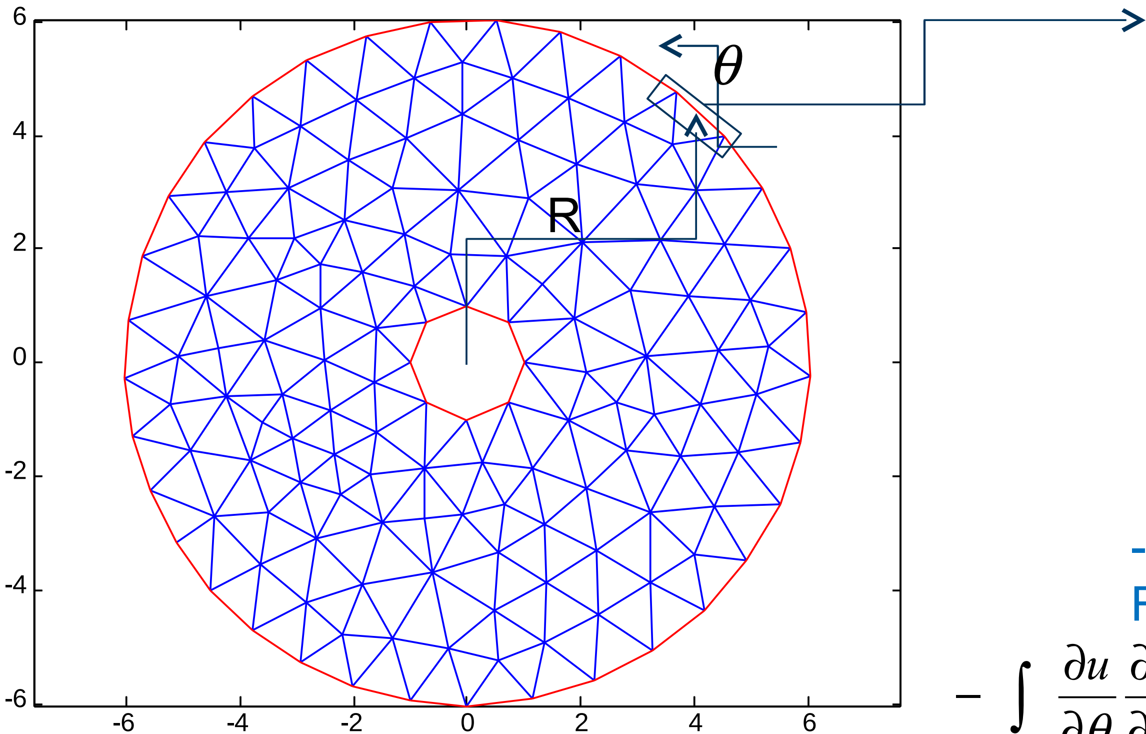
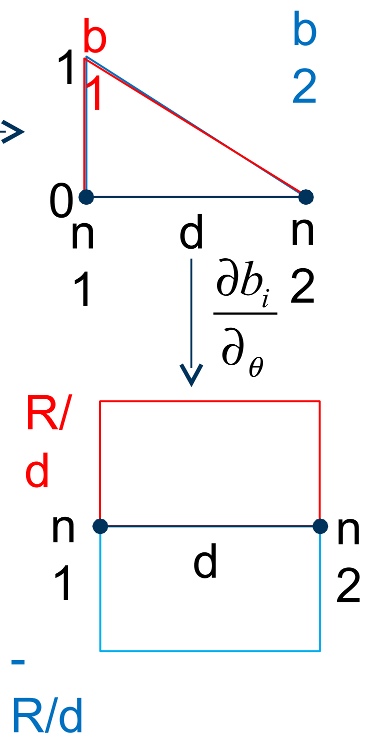
Higher order terms of BGT-2 and DtN-2

- $\int_{\Gamma_R} \frac{\partial^2 u}{\partial \theta^2} \bar{v} dS = -\int_{\Gamma_R} \frac{\partial u}{\partial \theta} \frac{\partial \bar{v}}{\partial \theta} dS$ can be inserted likewise.





$$-\int_{h\in\Gamma_R} \frac{\partial u}{\partial \theta} \frac{\partial \bar{v}}{\partial \theta} dS = \begin{cases} -R^2/d & \text{if same nodes} \\ R^2/d & \text{if neighbor nodes} \end{cases}$$

Higher order terms of BGT-2 and DtN-2

- Use the locations from the Boundary Mass Matrix to build up the boundary stiffness matrix.

```
if any(q(:)) | any(isnan(q(:)))
  kk=1./d;
  if methflag==1 %bgt-2 implementation
      qd=(((Rout^(2))/(2*Rout*(1-1j*k*Rout)))*(q.*(ones(N*N,1)*kk))); % Diagonal values
  elseif methflag==2 %dtn-2 implementation
      kR=k*Rout;
      aa=-besselh(1,1,kR)/besselh(0,1,(kR));
      bb=((1/kR)*besselh(1,1,kR)-besselh(2,1,kR))/besselh(1,1,kR);
      kR=k*(aa-bb);
      qd=(Rout^(2))*(kR)*(q.*(ones(N*N,1)*kk)); % Diagonal values
  end
  qod=-qd; % Off diagonal values
  m=1;
  for l=1:N,
    for k=1:N,
      Qs=Qs+sparse(ie1+(k-1)*np,ie2+(l-1)*np,qod(m,:),N*np,N*np);
      Qs=Qs+sparse(ie2+(k-1)*np,ie1+(l-1)*np,qod(m,:),N*np,N*np);
      Qs=Qs+sparse(ie1+(k-1)*np,ie1+(l-1)*np,qd(m,:),N*np,N*np);
      Qs=Qs+sparse(ie2+(k-1)*np,ie2+(l-1)*np,qd(m,:),N*np,N*np);
      m=m+1;
    end
  end
end
```

boundary stiffness matrix

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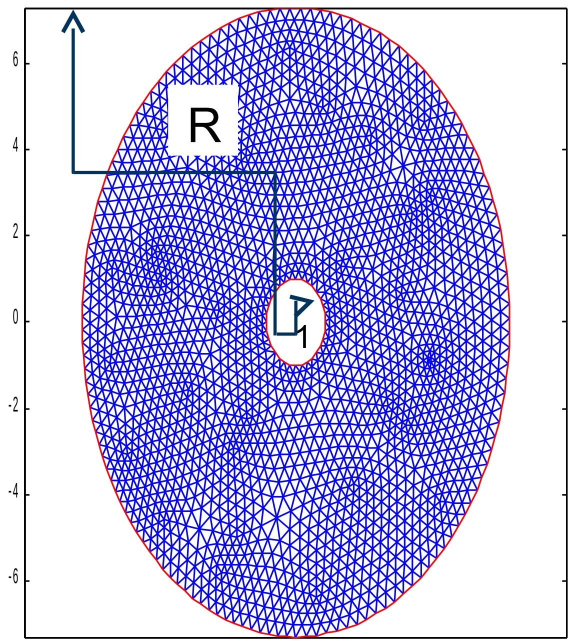
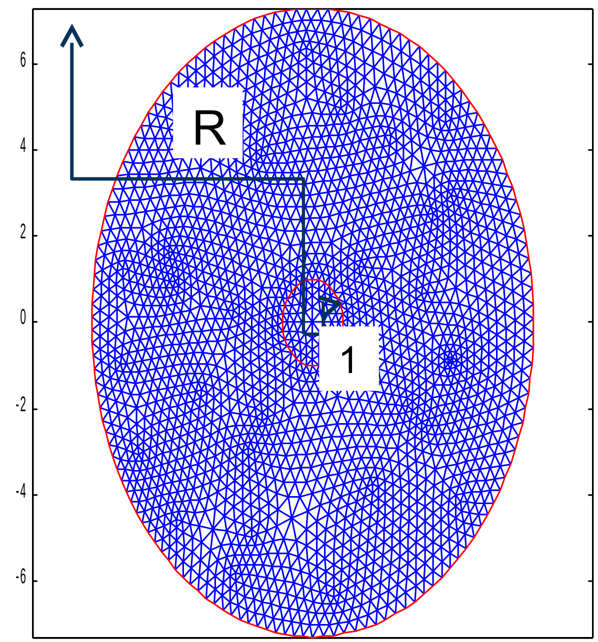
Numerical Experiment Setup

PEC (Acoustically Soft / Hard) case



Dielectric case



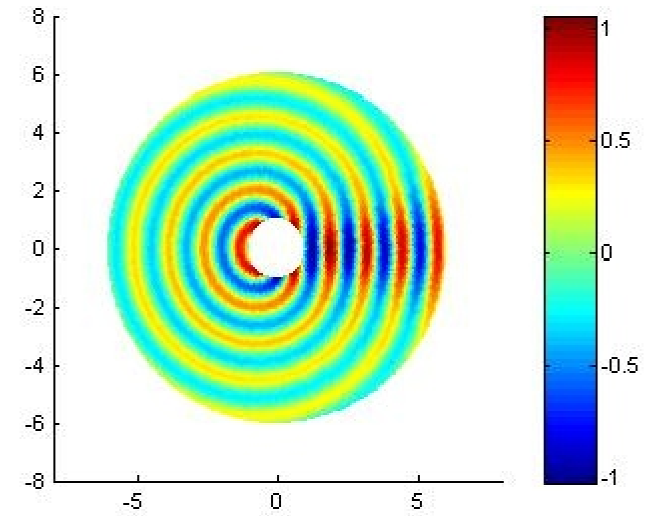
For both of the cases: $R := 1 + R_k \frac{2\pi}{k}$

Here: Rk=1 and k=1

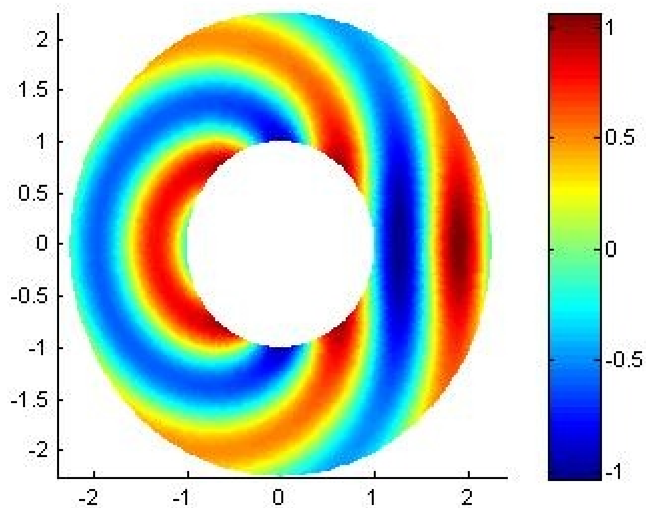
PEC, te case, k=5

L2 – error in the numerical volume

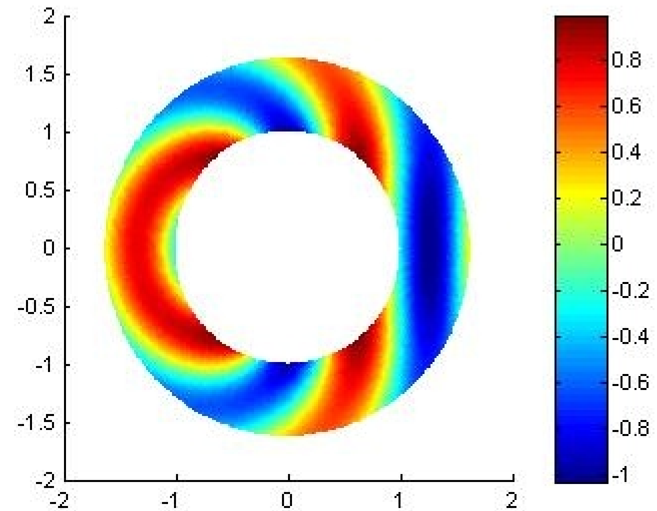
$$\mathrm{Re}\left(u^{scat}\right)$$



Rk=



Rk=1

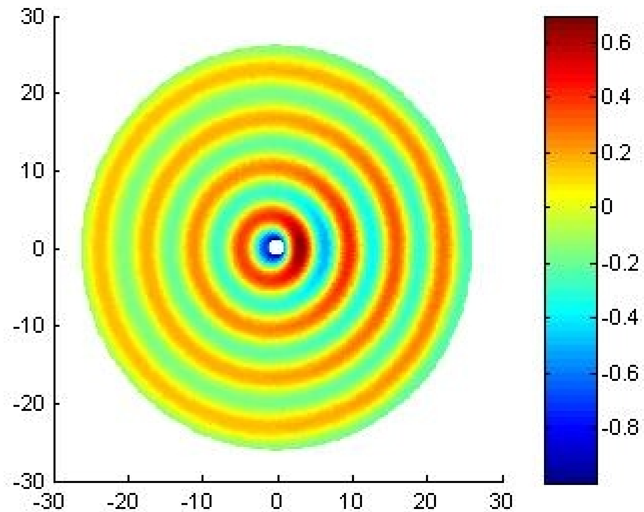


Rk=0.5

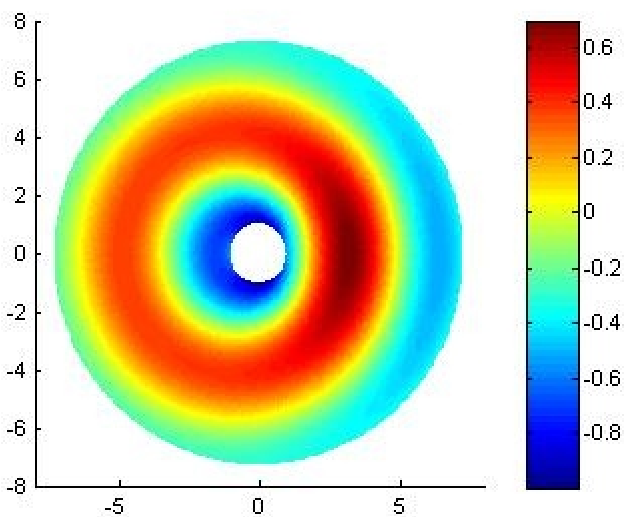
PEC, te case, k=1

L2 – error in the numerical volume

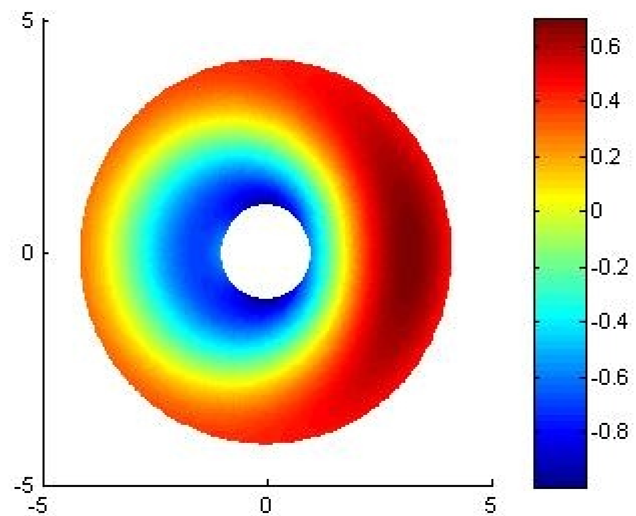
$\mathrm{Re}\left(u^{scat}\right)$



Rk=



Rk=1



Rk=0.5

PEC, te case, k=0.1

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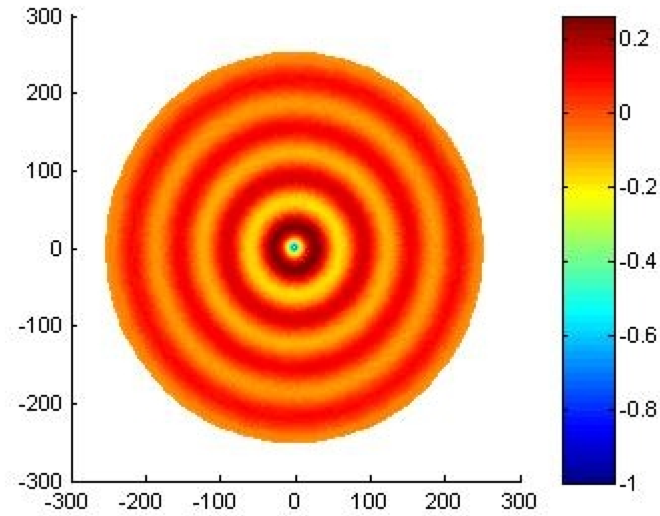
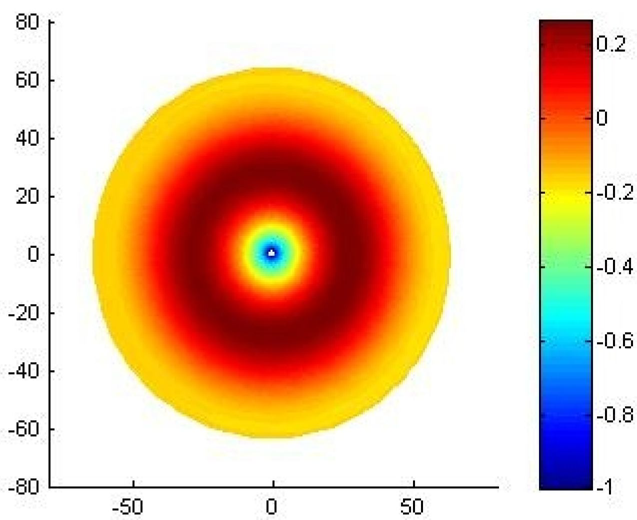
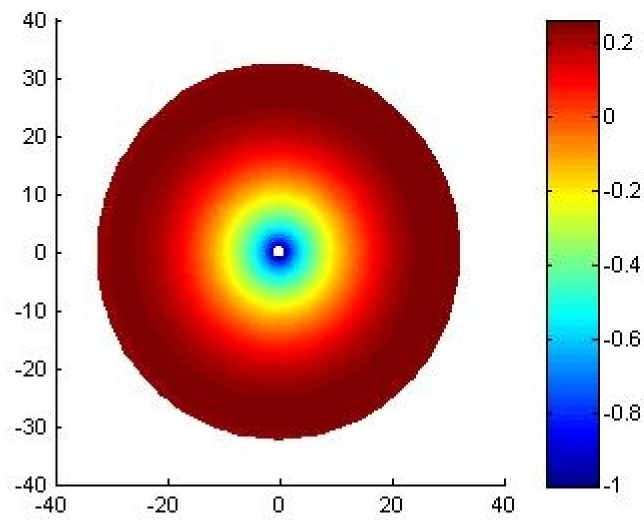


Rk=



Rk=
1



Rk=0.
5

dielectric, te case, k=1

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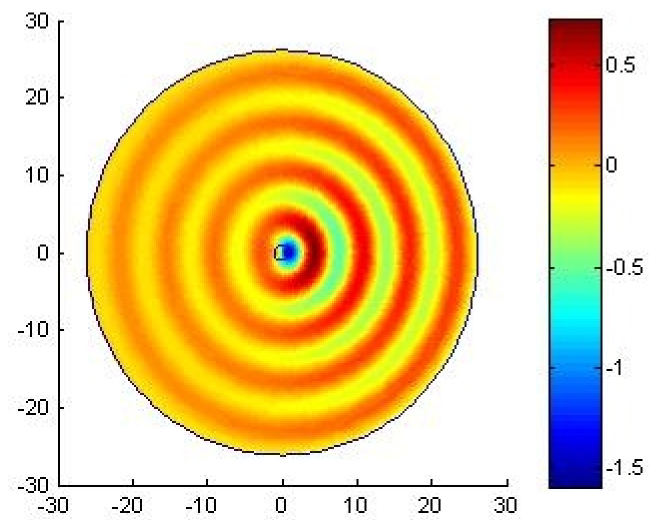
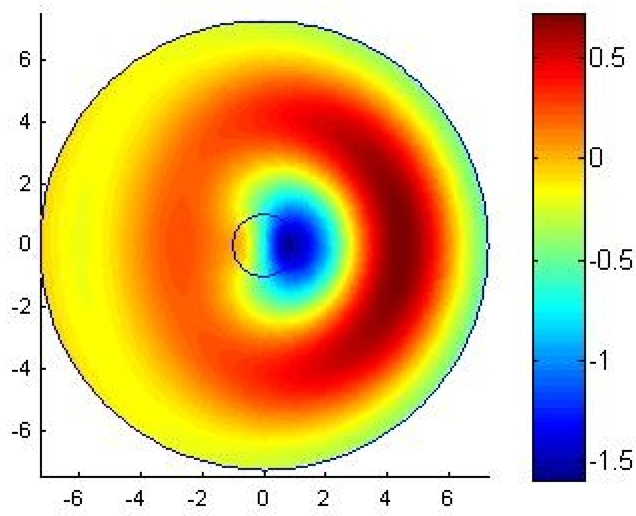
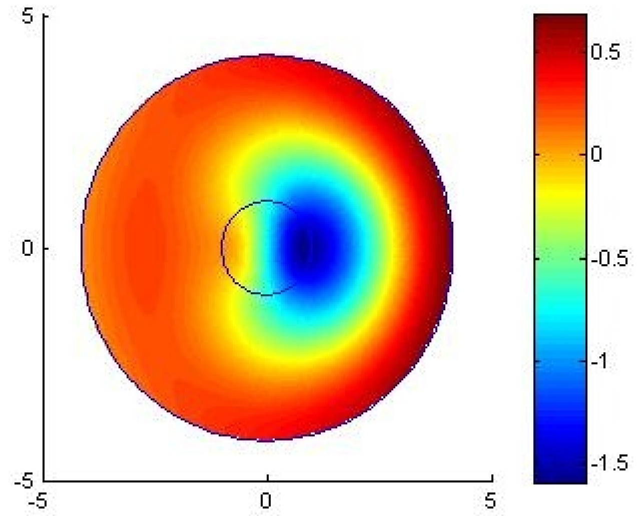
$\mathrm{Re}\left(u^{scat}\right)$



Rk=



Rk=1



Rk=0.5

dielectric, te case, k=0.1

L2 – error in the numerical volume

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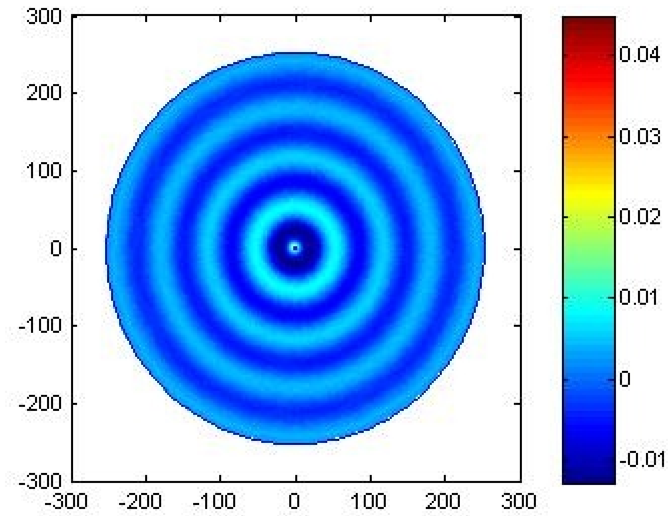
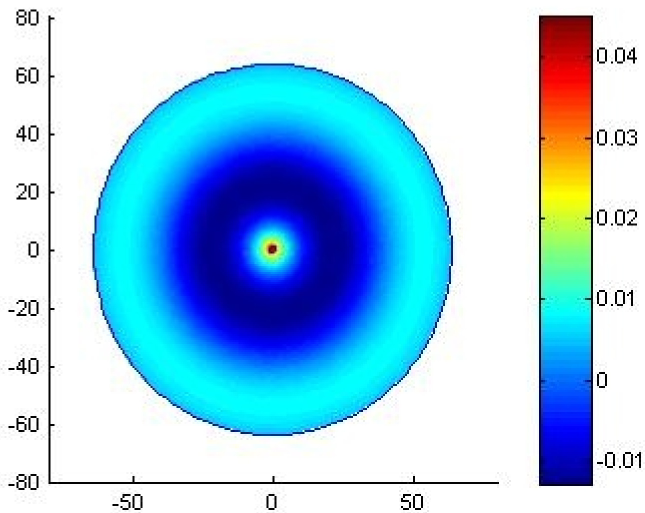
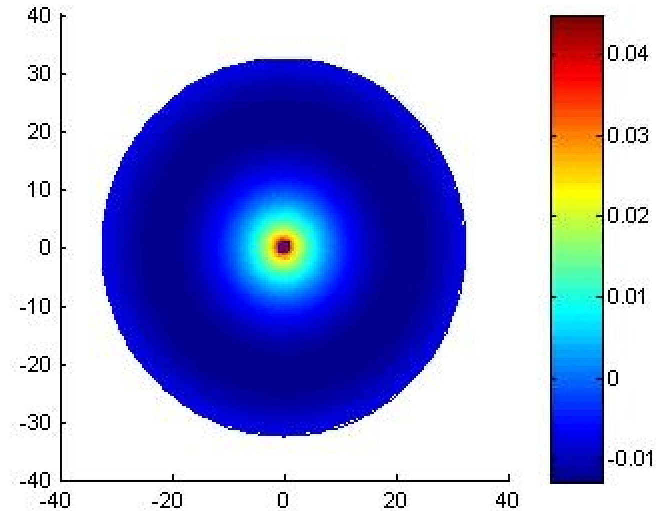


Rk=



Rk= 1



Rk=0.5

Discussion

- Why a big difference for small k and Rk values?
  - Comparison of the coefficients in the BCs.

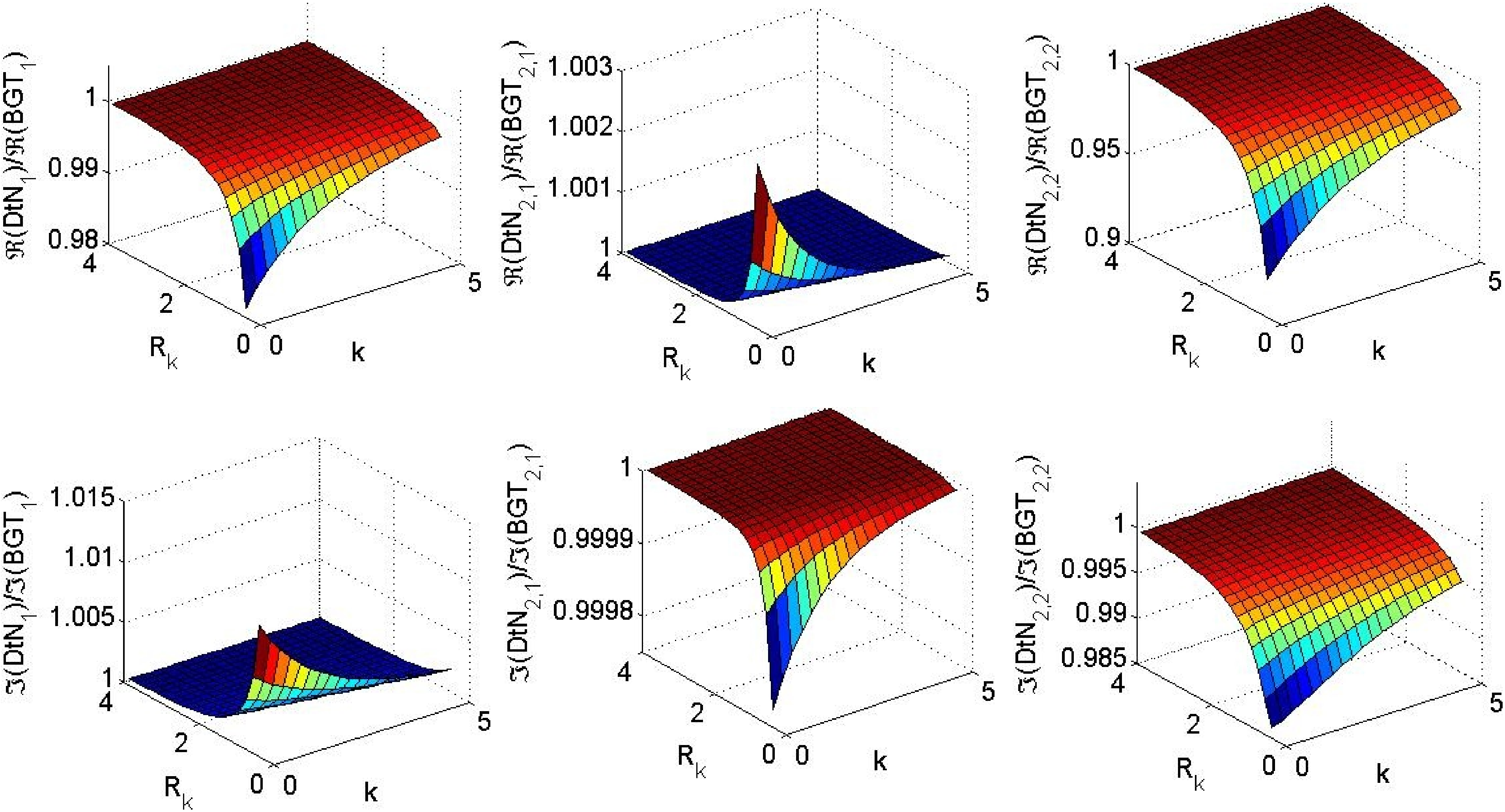
$$BGT-1: \frac{\partial u}{\partial r} = \underbrace{\left(ik - \frac{1}{2R} \right)}_{BGT_1} u$$

$$BGT-2: \frac{\partial u}{\partial r} = \underbrace{\left(ik - \frac{1}{2R} + \frac{1}{8R(1-ikR)} \right)}_{BGT_{2,1}} u + \underbrace{\frac{1}{2R(1-ikR)}}_{BGT_{2,2}} \frac{\partial^2 u}{\partial \theta^2}$$

$$DtN-1: \frac{\partial u}{\partial r} = \underbrace{k \frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)}}_{DtN_1} u$$

$$DtN-2: \frac{\partial u}{\partial r} = \underbrace{\left(k \frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)} \right)}_{DtN_{2,1}} u + \underbrace{k \left(\frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)} - \frac{H_1^{(1)'}(kR)}{H_1^{(1)}(kR)} \right)}_{DtN_{2,2}} \frac{\partial^2 u}{\partial \theta^2}$$

Comparison of constants



Conclusions

- Local ABCs are analyzed and implemented in a FEM scheme with linear basis functions.
  - Various numerical experiments performed for the dielectric and PEC scattering problem.
  - It is numerically verified that for low k and Rk values, DtN BCs are better than BGT BCs.