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Project: FEM/BEM coupling

We are interested in two-dimensional wave propagation problems occurring in linear acoustics and electromagnetism. To cover most of them, we set the following framework:

Consider an impinging plane-wave u^{inc} with wave-vector $\mathbf{k} = (k\cos\vartheta, k\sin\vartheta)$ scattered from a bounded object $\Omega \in \mathbb{R}^2$ whose closed boundary is Γ. We also introduce the unbounded complement domain $\Omega^c := \mathbb{R}^2 \setminus \bar{\Omega}$. The total wave u^{tot} can described as the sum

$$u^{\text{tot}} \;=\; u^{\text{inc}} + u \tag{1}$$

where u is the scattered wave. The total wave satisfies the homogeneous Helmholtz-type equation

$$-\operatorname{div} \alpha(\mathbf{x})\nabla u^{\text{tot}}(\mathbf{x}) \;-\; k^2\beta(\mathbf{x})u^{\text{tot}}(\mathbf{x}) \;=\; 0 \qquad \mathbf{x} \in \Omega^c \cup \Omega \tag{2}$$

where the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ lie in $L^\infty(\mathbb{R}^2)$, may vary in Ω but have constant values in Ω^c. Specifically,

$$\alpha(\mathbf{x}) \equiv \alpha_0 \qquad \text{and} \qquad \beta(\mathbf{x}) \equiv \beta_0 \qquad \text{for } \mathbf{x} \in \Omega^c. \tag{3}$$

with α_0, β_0 positive and real. The impinging plane wave u^{inc} satisfies the homogeneous Helmholtz-type equation in Ω^c. As seen in course, the above is problem is not well-defined unless boundary conditions and radiation conditions for the scattered field are included. These last are given by the Sommerfeld condition

$$\lim_{r\to\infty} \sqrt{r}\left|\frac{\partial u}{\partial r} - \imath k u\right| \;=\; 0 \tag{4}$$

We state the following different boundary conditions:

1. **Perfect conductor or acoustic hard material**

   Here, the trace of the total wave on Γ is equal to zero, i.e. if γ^c is the trace operator taken from the interior of Ω^c, then $\gamma u^{\text{tot}} \equiv 0$. Thus, the scattered wave must satisfy the Dirichlet condition:

   $$\gamma^c u \;=\; -\gamma^c u^{\text{inc}} \qquad \text{in } \Gamma. \tag{5}$$

   We shall denote by γ the trace taken within Ω.

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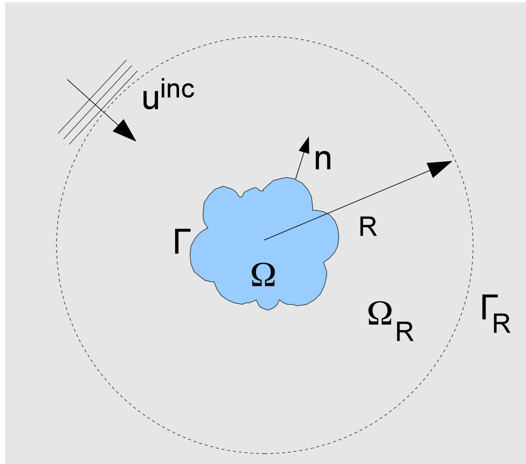

Abbildung 1: Geometry, notation and normal orientations.

2. Transmission conditions

In this case, we have to compute the field not only in Ω^c but in Ω. Hence, we can write the restrictions of the scattered field

$$u \;= u|_{\Omega} \qquad \text{and} \qquad u_c \;= u|_{\Omega^c} \tag{6}$$

and equivalently for u^{inc} and u^{tot}. These are linked through the so-called *transmission conditions*. Define the jump operator over Γ:

$$[\gamma u]_\Gamma \;:=\; \gamma^c u_c - \gamma u \qquad \text{on } \Gamma. \tag{7}$$

The first transmission condition is a null jump of the Dirichlet trace for the total field, i.e.

$$[\gamma u^{\text{tot}}]_\Gamma \;=\; 0 \qquad\qquad \text{Dirichlet jump} \tag{8}$$

We now require conditions on the normal derivatives:

$$\frac{\partial u}{\partial n} := \mathbf{n} \cdot \nabla u$$

with unit normal vectors $\mathbf{n}(x)$ pointing outwards Ω on the boundary Γ. Viewed from Ω^c the normal has opposite sign so for convenience we set $\mathbf{n}$ as pointing inside Ω^c in any case. Depending on the physics considered, we have

a) *TE polarization.* The transmission condition is given by normal continuity of the displacement vectors:

$$\left[\gamma\left(\alpha\frac{\partial u^{\text{tot}}}{\partial n}\right)\right]_\Gamma \;=\; \gamma^c\left(\alpha\frac{\partial u^{\text{tot}}}{\partial n}\right) - \gamma\left(\alpha\frac{\partial u^{\text{tot}}}{\partial n}\right) \;=\; 0. \tag{9}$$

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b) *TM waves and Acoustics.* Since in this case $\alpha \equiv 1$ everywhere, we have

$$\left[\gamma\left(\frac{\partial u^{\text{tot}}}{\partial n}\right)\right]_\Gamma = \gamma^c\left(\frac{\partial u^{\text{tot}}}{\partial n}\right) - \gamma\left(\frac{\partial u^{\text{tot}}}{\partial n}\right)_\Gamma = 0. \qquad (10)$$

Tasks

1. Consider a bounded **perfectly conducting** (or **acoustically soft**) scatterer Ω.

a) Write down the continuous variational formulation for the scattered field u using (2) with boundary condition (5) and radiation conditions (4) for $\alpha(x) = \beta(x) = 1$ everywhere. State the functional framework explicitly. From results presented in class, show that the solution u exists and is unique.

b) Introduce an artificial boundary Γ_R given by a ball centered at zero of radius R, denoted B_R enclosing Ω (see Figure 1). Modify accordingly the above variational form.

(i) State *Johnson's-Nédélec formulation* [6, 2] over Γ_R and show how in this case the BEM intervenes in the continuous case for modeling the solution in B_R^c.

(ii) Repeat the same for the *symmetric formulation* [1].

c) Write down the discrete Galerkin form of part (b) using the FEM for describing the truncated domain $\Omega_R^c := \Omega^c \cap B_R$ and the BEM (for both formulations).

(i) State discrete spaces and assume first-order polynomials ($\mathbb{P}_1$) for the volume approximation.

(ii) Accordingly construct the approximations by the BEM.

(iii) Present the matrix form of the problem.

d) Implement the FEM/BEM using MATLAB and its PDE toolbox to obtain stiffness and mass matrices. Contributions coming from integral operators are obtained through the MEX-Fortran library written by A. Bendali. Consider Ω as the ball of radius one, B_1, and set the artificial boundary at a radius R.

e) Let $\vartheta = 0$. The exact solution of the problem with radiation conditions (4) is given in terms of an infinite series of Hankel functions. Compare the relative error in maximum norm between your numerical solution and the exact solution (a MATLAB script with a series representation of the exact solution can be downloaded form the course home page). Run your code for $k = 0.001, 0.1, 1, 5, 10$

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and study the convergence for increasing mesh refinements for a fixed R and for increasing values of R. Propose how the BEM parameters should be chosen for each formulation.

2. Now let Ω be a bounded **dielectric scatterer**:

a) Write down the continuous variational formulation for the scattered field u using (2) with boundary condition (5) and radiation conditions (4). Using the results from the course show that the solution u exists and is unique.

b) Introduce an artificial boundary Γ_R given by a ball centered at zero of radius R, denoted B_R enclosing Ω (see Fig. 1). Modify accordingly the above variational form. State how the different BEM formulations intervene in the continuous case.

c) Show how the discrete Galerkin form (using FEM for describing the truncated domain $\Omega_R^c := \Omega^c \cap B_R$) and the BEM need to be modified for both formulations. Present the matrix form of the problem.

d) Implement the method computationally using MATLAB and its PDE toolbox to obtain matrices stiffness and mass matrices. Consider Ω as the ball of radius one, B_1, and set the artificial boundary at a radius R.

e) Take $\vartheta = 0$ and let $\alpha = 1$ in $\Omega^c \cup \Omega$, $\beta = 1$ for $\mathbf{x} \in \Omega$ and $\beta = 4$ for $\mathbf{x} \in \Omega^c$. Run your code for $k = 0.1, 1, 5$ and study the convergence for increasing mesh refinements for a fixed R and for increasing values of R (MATLAB mat-files with accurate solutions can be downloaded form the course home page).

f) Take $\vartheta = 0$ and let $\beta = 1$ in $\Omega^c \cup \Omega$, $\alpha = 1$ for $\mathbf{x} \in \Omega$ and $\alpha = 1/4$ for $\mathbf{x} \in \Omega^c$. Run your code for $k = 0.1, 1, 5$ and study the convergence for increasing mesh refinements for a fixed R and for increasing values of R (MATLAB mat-files with accurate solutions can be downloaded form the course home page).

g) Use a rectangle with side lengths $l_x = 5$ and $l_y = 1$ to study the numerical efficiency of the method for elongated obstacles. Take $\vartheta = 0$ and let $\alpha = 1$ in $\Omega^c \cup \Omega$, $\beta = 1$ for $\mathbf{x} \in \Omega$ and $\beta = 4$ for $\mathbf{x} \in \Omega^c$. Run your code for $k = 1$, and study the convergence for increasing mesh refinements for a fixed R and for increasing values of R (MATLAB mat-files with an accurate solution can be downloaded form the course home page).

Siehe nächstes Blatt!

Literatur

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