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Project: FEM with PML

We are interested in two-dimensional wave propagation problems occurring in linear acoustics and electromagnetism. To cover most of them, we set the following framework:

Consider an impinging plane-wave u^{inc} with wave-vector $\mathbf{k} = (k\cos\vartheta, k\sin\vartheta)$ scattered from a bounded object $\Omega \in \mathbb{R}^2$ whose closed boundary is Γ. We also introduce the unbounded complement domain $\Omega^c := \mathbb{R}^2 \setminus \bar{\Omega}$. The total wave u^{tot} can described as the sum

$$u^{\text{tot}} \;=\; u^{\text{inc}} + u \tag{1}$$

where u is the scattered wave. The total wave satisfies the homogeneous Helmholtz-type equation

$$-\operatorname{div}\alpha(\mathbf{x})\nabla u^{\text{tot}}(\mathbf{x}) \;-\; k^2\beta(\mathbf{x})u^{\text{tot}}(\mathbf{x}) \;=\; 0 \qquad \mathbf{x} \in \Omega^c \cup \Omega \tag{2}$$

where the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ lie in $L^\infty(\mathbb{R}^2)$, may vary in Ω but have constant values in Ω^c. Specifically,

$$\alpha(\mathbf{x}) \equiv \alpha_0 \qquad \text{and} \qquad \beta(\mathbf{x}) \equiv \beta_0 \qquad \text{for } \mathbf{x} \in \Omega^c. \tag{3}$$

with α_0, β_0 positive and real. The impinging plane wave u^{inc} satisfies the homogeneous Helmholtz-type equation in Ω^c. As seen in course, the above is problem is not well-defined unless boundary conditions and radiation conditions for the scattered field are included. These last are given by the Sommerfeld condition

$$\lim_{r\to\infty} \sqrt{r}\left|\frac{\partial u}{\partial r} - \imath k u\right| \;=\; 0 \tag{4}$$

We state the following different boundary conditions:

1. **Perfect conductor or acoustic hard material**

   Here, the trace of the total wave on Γ is equal to zero, i.e. if γ^c is the trace operator taken from the interior of Ω^c, then $\gamma u^{\text{tot}} \equiv 0$. Thus, the scattered wave must satisfy the Dirichlet condition:

   $$\gamma^c u \;=\; -\gamma^c u^{\text{inc}} \qquad \text{in } \Gamma. \tag{5}$$

   We shall denote by γ the trace taken within Ω.

Bitte wenden!

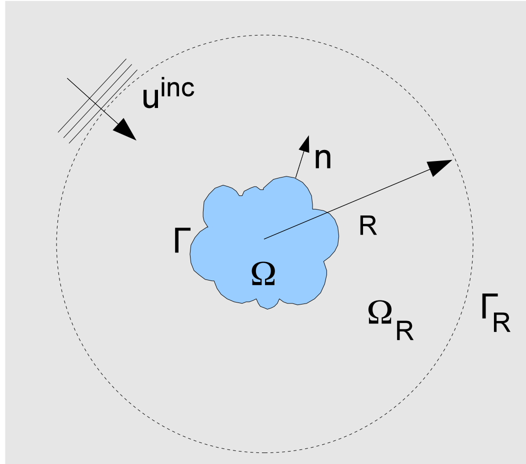

Abbildung 1: Geometry, notation and normal orientations.

2. Transmission conditions

In this case, we have to compute the field not only in Ω^c but in Ω. Hence, we can write the restrictions of the scattered field

$$u \;= u|_{\Omega} \qquad \text{and} \qquad u_c \;= u|_{\Omega^c} \tag{6}$$

and equivalently for u^{inc} and u^{tot}. These are linked through the so-called *transmission conditions*. Define the jump operator over Γ:

$$[\gamma u]_{\Gamma} \;:=\; \gamma^c u_c - \gamma u \qquad \text{on } \Gamma. \tag{7}$$

The first transmission condition is a null jump of the Dirichlet trace for the total field, i.e.

$$[\gamma u^{\text{tot}}]_{\Gamma} \;=\; 0 \qquad\qquad \text{Dirichlet jump} \tag{8}$$

We now require conditions on the normal derivatives:

$$\frac{\partial u}{\partial n} := \mathbf{n} \cdot \nabla u$$

with unit normal vectors $\mathbf{n}(x)$ pointing outwards Ω on the boundary Γ. Viewed from Ω^c the normal has opposite sign so for convenience we set $\mathbf{n}$ as pointing inside Ω^c in any case. Depending on the physics considered, we have

a) *TE polarization.* The transmission condition is given by normal continuity of the displacement vectors:

$$\left[\gamma\left(\alpha\frac{\partial u^{\text{tot}}}{\partial n}\right)\right]_{\Gamma} \;=\; \gamma^c\left(\alpha\frac{\partial u^{\text{tot}}}{\partial n}\right) - \gamma\left(\alpha\frac{\partial u^{\text{tot}}}{\partial n}\right) \;=\; 0. \tag{9}$$

Siehe nächstes Blatt!

b) *TM waves and Acoustics.* Since in this case $\alpha \equiv 1$ everywhere, we have

$$\left[\gamma\left(\frac{\partial u^{\text{tot}}}{\partial n}\right)\right]_\Gamma = \gamma^c\left(\frac{\partial u^{\text{tot}}}{\partial n}\right) - \gamma\left(\frac{\partial u^{\text{tot}}}{\partial n}\right)_\Gamma = 0. \tag{10}$$

Tasks

1. Consider a bounded **perfectly conducting** (or **acoustically soft**) scatterer Ω.

a) Write down the continuous variational formulation for the scattered field u using (2) with boundary condition (5) and radiation conditions (4). State the functional framework explicitly. From results presented in class, show that the solution u exists and is unique.

b) Introduce an artificial boundary Γ_R given by a ball centered at zero of radius R, denoted B_R enclosing Ω; See Figure 1. Modify accordingly the above variational form to include the PML in curvilinear coordinates [1].

c) Write down the discrete Galerkin form of part (b) using FEM for describing the truncated domain $\Omega_R^c := \Omega^c \cap B_R$ and PML. State discrete spaces and assume first-order polynomials ($\mathbb{P}_1$) for the volume approximation. Accordingly construct the approximations by the FEM/PML. Present the matrix form of the problem.

d) Implement the PML using MATLAB (or an equivalent software) to obtain stiffness and mass matrices. Consider Ω as the ball of radius one, B_1, and set the artificial boundary at a radius R.

e) Let $\vartheta = 0$, $\alpha = 1$, $\beta = 1$ in Ω^c. The exact solution of the problem with radiation conditions (4) is given in terms of an infinite series of Hankel functions. Compare the relative error in maximum norm between your numerical solution and the exact solution (a MATLAB script with a series representation of the exact solution can be downloaded form the course home page). Run your code for $k = 0.001, 0.1, 1, 5, 10$ and study the convergence for increasing mesh refinements for a fixed R and for increasing values of R and for increasing thickness of the PML layer.

2. Now let Ω be a bounded **dielectric scatterer**:

a) Write down the continuous variational formulation for the scattered field u using (2) with boundary condition (5) and radiation conditions (4). Using the results from the course show that the solution u exists and is unique.

Bitte wenden!

b) Introduce an artificial boundary Γ_R given by a ball centered at zero of radius R, denoted B_R enclosing Ω; See Figure 1. Modify accordingly the above variational form.

c) Show how the discrete Galerkin form (using FEM for describing the truncated domain $\Omega_R^c := \Omega^c \cap B_R$) and the FEM/PML needs to be modified. Present the matrix form of the problem.

d) Implement the PML using MATLAB (or an equivalent software) to obtain matrices stiffness and mass matrices. Consider Ω as the ball of radius one, B_1, and set the artificial boundary at a radius R.

e) Take $\vartheta = 0$ and let $\alpha = 1$ in $\Omega^c \cup \Omega$, $\beta = 1$ for $\mathbf{x} \in \Omega$ and $\beta = 4$ for $\mathbf{x} \in \Omega^c$. Run your code for $k = 0.1, 1, 5$ and study the convergence for increasing mesh refinements for different types of PML; see (MATLAB mat-files with accurate solutions can be downloaded form the course home page).

f) Take $\vartheta = 0$ and let $\beta = 1$ in $\Omega^c \cup \Omega$, $\alpha = 1$ for $\mathbf{x} \in \Omega$ and $\alpha = 1/4$ for $\mathbf{x} \in \Omega^c$. Run your code for $k = 0.1, 1, 5$ and study the convergence for increasing mesh refinements for a fixed R and for increasing values of R and for increasing thickness of the PML layer (MATLAB mat-files with accurate solutions can be downloaded form the course home page).

g) Implement PML in Cartesian coordinates [1, 2]. Use a rectangle with side lengths $l_x = 5$ and $l_y = 1$ to study the numerical efficiency of the new condition compared to the circular PML for elongated obstacles. Take $\vartheta = 0$ and let $\alpha = 1$ in $\Omega^c \cup \Omega$, $\beta = 1$ for $\mathbf{x} \in \Omega$ and $\beta = 4$ for $\mathbf{x} \in \Omega^c$. Run your code for $k = 1$, and study the convergence for increasing mesh refinements and for the different type of PML used in [1, 2] (a MATLAB mat-file with an accurate solution can be downloaded form the course home page).

Literatur

[1] F. Collino and P. Monk, *The Perfectly Matched Layer in Curvilinear Coordinates*, SIAM J. Sci. Comput. 19 (6) (1998) 2061-2090.

[2] A. Bermudez and L. Hervella-Nieto and A. Prieto and R. Rodriguez, *An optimal perfectly matched layer with unbounded absorbing function for time-harmonic acoustic scattering problems*, SIAM J. Sci. Comput. 30 (1) (2007) 312-338.