Sol. Exercise V

1)

a) $$\int_{\Omega_R^c} \left(-\Delta \bar{u}\, u - \bar{k}^2 |u|^2\right) dx = \|\nabla u\|^2_{L^2(\Omega_R^c)} - \bar{k}^2 \|u\|^2_{L^2(\Omega_R^c)}$$
$$= \int_{\partial\Omega_R^c} \frac{\partial \bar{u}}{\partial n}\, u \, ds = \int_{\partial B_R} \frac{\partial \bar{u}}{\partial r}\, u\, ds - \int_{\partial\Omega^c} \frac{\partial \bar{u}}{\partial n}\, u\, ds$$

b) Multiply by k and take imaginary parts:

$$\operatorname{Im}(k)\left(\|\nabla u\|^2_{L^2(\Omega_R^c)} + |k|^2 \|u\|_{L^2(\Omega_R^c)}\right)$$
$$= -\operatorname{Im}\left(\int_\Gamma k \frac{\partial \bar{u}}{\partial n} u\, ds\right) + \operatorname{Im}\left(\int_{\partial B_R} k \frac{\partial \bar{u}}{\partial r} u\, ds\right)$$

using (4),

$$= -\int_\Gamma \operatorname{Im}\left(k \frac{\partial \bar{u}}{\partial n} u\right) ds + \frac{1}{2}\int_{\partial B_R} \left|\frac{\partial u}{\partial r} - iku\right|^2 ds \qquad (*)$$
$$- \frac{1}{2}\int_{\partial B_R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + |k|^2 |u|^2\right) ds$$

Let $R \to \infty$. Since $\operatorname{Im}(k) > 0$, Sommerfeld integral goes to zero, and so by the condition on the integral in Γ, $u = 0$ over Ω^c.

c) Now the L.H.S is zero. Then,

$$0 = -\frac{1}{2}\int_{\partial B_R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + |k|^2 |u|^2\right) ds - \int_\Gamma \operatorname{Im}\left(k \frac{\partial \bar{u}}{\partial n} u\right) ds$$

$\Rightarrow \int |u|^2 ds = 0 \Rightarrow$ Rellich theorem applies $\Rightarrow u = 0 \in \Omega^c$.

2] a) $\displaystyle\int_{\partial B_R} u \frac{\partial \bar{u}}{\partial r}\, ds = \int_{\partial\Omega} u \frac{\partial \bar{u}}{\partial n}\, ds - k^2 \int_{\Omega_R^c} |u|^2\, dx + \int_{\Omega_R^c} |\nabla u|^2\, dx$

Take imaginary parts and limit $R \to \infty$

$$\int_{\partial B_R} \operatorname{Im}\left(u \frac{\partial \bar{u}}{\partial r}\right) ds = \int_{\Gamma} \operatorname{Im}\left(u \frac{\partial \bar{u}}{\partial n}\right) ds$$

and using (*)

$$\lim_{R\to\infty} \int_{\partial B_R} \left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2 |u|^2\right) ds = -2k \operatorname{Im} \int_{\Gamma} u \frac{\partial \bar{u}}{\partial n}\, ds$$

Since, the l.H.s is composed of positive terms and the r.h.s is bounded. The terms are individually bounded. $\Rightarrow \displaystyle\int_{S_R} |u|^2\, ds = O(1)$ as $R \to \infty$.

b)

$$|I_1| \le \left(\int_{S_R} |u|^2 ds\right)^{1/2} \cdot \Bigg(\int_{S_R} \underbrace{\left|\frac{\partial G}{\partial n} - ikG\right|^2 ds(y)}_{O\left(\frac{1}{r^2}\right)}\Bigg)^{1/2} \to 0$$

as $R \to \infty$.

$$|I_2| \le \Bigg(\int_{S_R} \underbrace{|G(x,y)|^2}_{O\left(\frac{1}{r}\right)^2} d\sigma(y)\Bigg)^{1/2} \cdot \Bigg(\int_{S_R} \underbrace{\left|\frac{\partial u}{\partial r} - iku\right|^2}_{O\left(\frac{1}{r^2}\right)} ds(y)\Bigg)^{1/2} \xrightarrow[R\to\infty]{} 0.$$

c) From Green's second inequality,

$$\int_{\Omega_R^c} (-\Delta u - k^2 u)\, G(x,y) + (\Delta G + k^2 G)\, u =$$

$$= \int_{\partial \Omega^c_R} \underbrace{\left(u \frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G \right)}_{F(u,G)} ds = \int_{\partial B_R} + \int_{\partial \Omega}$$

From the representation formula in a bounded domain.

$$u(x) = \int_{\partial B_R} F(u,G) + \int_{\partial \Omega} F(u,G)$$

But

$$\int_{\partial B_R} F(u,G) = I_1 - I_2 \to 0 \quad \text{as } R \to \infty.$$

3] a) Taylor $n = 1$, $0 \le r \le r_0 < 1$

$$\frac{1}{\sqrt{1 - re^{\pm i\theta}}} = \frac{1}{\sqrt{1-\alpha}} = 1 + \left(- \frac{1}{2} \frac{(-1)}{(\sqrt{1-\alpha})^3} \right) \Bigg|_{\alpha=0} \alpha$$

$$= 1 + \frac{1}{2} \alpha$$

$\underline{n = 2}$

$$\frac{d}{d\alpha} \left(\frac{1}{2} (1-\alpha)^{-3/2} \right) = \frac{1}{2} \left(-\frac{3}{2} \right) (1-\alpha)^{-5/2} \cdot (-1)$$

$$\to \frac{d^2}{d\alpha^2} : \frac{1}{2} \cdot \frac{3}{2} (1-\alpha)^{-5/2} \Rightarrow \frac{1}{\sqrt{1-\alpha}} = 1 + \frac{1}{2}\alpha + \frac{3}{4}\alpha^2$$

Induction:

$$\frac{d^n}{d\alpha^n} (1-\alpha)^{-1/2} = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} (1-\alpha)^{-\frac{(2n+1)}{2}}$$

$$\frac{d}{d\alpha} \left(\frac{d^n}{d\alpha^n} (1-\alpha)^{-1/2} \right) = \frac{1 \cdots (2n-1)}{2 \cdots 2n} \cdot -\frac{(2n+1)}{2} \cdot (-1) \cdot (1-\alpha)^{-\frac{(2n+3)}{2}}$$

$$= \frac{d^{n+1}}{d\alpha^{n+1}}$$

Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2n+1}{2(n+1)}\cdot r < 1$$

$\Rightarrow$ converges absolutely

M - test:

$$\sum_{n=1}^{\infty}{}' \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} r^n e^{\pm in\theta} \le \sum_{n=1}^{\infty}{}' \left|\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right| r^n$$

$\Rightarrow$ uniformly convergent

$$\le \sum_{n=1}^{\infty}\left|\frac{1\cdot 2\cdots}{\cdots}\right| r_0^n$$

$r_0 \in (0,1)$

b) Notice that

$$\frac{1}{\sqrt{1-re^{i\theta}}}\cdot\frac{1}{\sqrt{1-re^{-i\theta}}} = \frac{1}{\sqrt{1-2r\cos\theta+r^2}} \qquad t=\cos\theta$$

Since the series are convergent (uniformly and absolutely) take $\theta=0$. Then, the geometric series is a majorant for (9) $\Rightarrow$ $|P_n(t)| \le 1 \quad -1\le t\le 1$

c) Differentiating (9) yields w.r.t. r

$$-\frac{1}{2}\frac{(-2t+2r)}{(1-2tr+r^2)^{3/2}} = \sum_{n=0}^{\infty}{}' n P_n(t) r^{n-1} \Big/ 1-2tr+r^2$$

$$\frac{t-r}{\sqrt{1-2tr+r^2}} = \sum_{n=0}^{\infty}{}' n P_n(t) r^{n-1}(1-2tr+r^2)$$

$$\sum_{n=0}^{\infty}{}' P_n(t)\, r^n (t-r) = \sum_{n=0}^{\infty}{}' n P_n(t) r^{n-1}(1-2tr+r^2)$$

Coefficient: r^n

$$P_n t^{2n} - P_{n+1} = nP_n(-2t) + nP_{n-1} + nP_{n+1}$$

$$\Rightarrow \quad 0 = (n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}$$

$n=0,1,\dots$

d)
$$\int_{-1}^{1} \frac{dt}{1-2tr+r^2} = \int_{-1}^{1}\left(\sum_{n=0}^{\infty} P_n(t) r^n\right)^2 dt$$

By orthogonality of P_n (spherical harmonics)

$$\int_{-1}^{1} g^2\,dt = \sum_{n=0}^{\infty}\int_{-1}^{1}[P_n(t)]^2\,dt\, r^{2n}$$

uniform convergence allows the exchange of integration/summation

On the other hand,

$$\int_{-1}^{1}\frac{dt}{1-2tr+r^2} = \frac{1}{r}\ln\frac{1+r}{1-r} = \sum_{n=0}^{\infty}\frac{2}{2n+1}r^{2n}$$

Then

$$\int_{-1}^{1}[P_n(t)]^2\,dt = \frac{2}{2n+1}.$$

4) a) Each harmonic polynomial is mapped again into homogeneous harmonic polynomials by Q orthogonal $\Rightarrow$ (15)

Now, since Q is orthogonal and Y_n^m are orthonormal

$$\int_{S^2} Y_n^m(Q\hat{x})\overline{Y_n^{m'}(Q\hat{x})}\,ds = \int_{S^2} Y_n^m(\hat{x})Y_n^{m'}(\hat{x})\,ds$$
$$= \delta_{mm'}$$

Hence, in $Y_n^m(Q\hat{x}) = \sum_{k=-n}^{n} a_{mk} Y_n^k(\hat{x})$ $\quad m=-n,\dots,n$

$A = (a_{mk})$ is orthogonal and

$$Y(Q\hat{x}, Q\hat{y}) = \sum_{m=-n}^{n}{}' \left(\sum_{k=-n}^{n} a_{mk} Y_n^k(\hat{x})\right)\left(\overline{\sum_l{}' a_{ml} Y_n^l(\hat{y})}\right)$$

$$= \sum_{k=-n}^{n}{}' Y_n^k(\hat{x})\overline{Y_n^k(\hat{y})} = Y(\hat{x},\hat{y})$$

$\Rightarrow$ $Y(\hat{x},\hat{y}) = f(\cos\theta)$.

b)

$$\sum_{m=-n}^{n}{}' Y_n^m(\hat{x})\overline{Y_n^m(\hat{y})} = a_n P_n(\cos\theta)$$

Set $\hat{x}=\hat{y} \Rightarrow \theta = 0, \cos\theta = 1$

$$\Rightarrow \sum_{m=-n}^{n}{}' |Y_n^m(\hat{x})|^2 = a_n P_n(1) = a_n \quad \Big/ \int_{S^2} ds$$

$$\Rightarrow 4\pi a_n = \sum_{m=-n}^{n}{}' \underbrace{\int_{S^2} |Y_n^m(\hat{x})|^2 \, ds(\hat{x})}_{=1 \text{ by definition}}$$

$$\Rightarrow \boxed{a_n = \frac{2n+1}{4\pi}}$$