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## On the Coupling of Boundary Integral and Finite Element Methods

By Claes Johnson and J. Claude Nedelec

**Abstract.** Let  $\Omega^c$  be the complementary of a bounded regular domain in  $\mathbb{R}^2$  of boundary  $\Gamma$ . We consider the problem

$$(1) \quad \begin{cases} \Delta u = f; & \text{in } \Omega^c, \\ u|_{\Gamma} = u_0, \end{cases}$$

where  $f$  has its support in a bounded subdomain  $\Omega_1$  of  $\Omega^c$ . Let  $\Gamma_2$  be the common boundary of  $\Omega_1$  and  $\Omega_2 = \Omega^c - \Omega_1$ . We solve the problem (1) by using an equivalent system of equations involving an integral equation on  $\Gamma_2$  coupled with the equation:

$$(2) \quad \begin{cases} \Delta u = f & \text{in } \Omega_1, \\ u|_{\Gamma} = u_0, \\ u|_{\Gamma_2} = \lambda. \end{cases}$$

We introduce a finite element approximation of Eq. (2) and of the integral equation and we prove optimal error estimates.

**Introduction.** The purpose of this note is to analyze a procedure obtained by coupling the boundary integral method (cf. [4], [5], [7], [8], [12]) and the usual finite element method. Such coupled procedures have been proposed by e.g. Silvester-Hsieh [10] and Zienkiewicz et al. [11] for the numerical solution of problems in unbounded domains. As a typical example let us consider a problem of the form

$$\begin{cases} Au = f & \text{in } \Omega^c, \\ u = u_0 & \text{on } \Gamma, \end{cases}$$

where  $\Omega$  is a bounded domain in the plane with boundary  $\Gamma$ ,  $\Omega^c$  is the unbounded complement of  $\Omega$ , and  $A$  is an elliptic differential operator. Let us further assume that  $\Omega^c$  can be divided into a bounded part  $\Omega_1$  and an unbounded part  $\Omega_2$ , with common boundary  $\Gamma_2$  (see Figure 1), so that  $f = 0$  in  $\Omega_2$  and  $A$  is linear and has constant coefficients in  $\Omega_2$  while  $A$  may be nonlinear or have variable coefficients in the bounded part  $\Omega_1$ . Then the unbounded part  $\Omega_2$  can be taken into account using an integral equation on the boundary  $\Gamma_2$ , and an approximate solution can be found using a conventional finite element discretization of  $\Omega_1$  together with a discretization along  $\Gamma_2$ . Below we shall analyze a model problem of this type.

For numerical experiments and references into the engineering literature on this subject, we refer to [11].

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1. **A Model Problem.** Let us consider the following exterior Dirichlet problem:

$$(1.1) \quad \begin{cases} -\Delta u = f & \text{in } \Omega^c, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^2$  with smooth boundary  $\Gamma$  and  $\Omega^c$  is the complement of  $\Omega \cup \Gamma$ . Let us assume that the support of  $f$  is bounded and that  $f \in L^2(\Omega^c)$ . It is known (see e.g. [3], [6]) that the problem (1.1) admits a unique solution  $u \in W^1(\Omega^c)$ , where

$$W^1(\Omega^c) = \{v: (1 + |x|^2)^{-1/2}(1 + \log\sqrt{1 + |x|^2})^{-1}v \in L^2(\Omega^c), \nabla v \in [L^2(\Omega^c)]^2\},$$

and that this solution has the following asymptotic behavior:

$$(1.2) \quad \begin{cases} u(x) = \alpha + O\left(\frac{1}{|x|}\right), & |x| \rightarrow \infty, \\ \nabla u(x) = O\left(\frac{1}{|x|^2}\right), & |x| \rightarrow \infty, \end{cases}$$

where  $\alpha$  is a constant.

Let now  $\Gamma_2$  be a smooth curve dividing  $\Omega^c$  into an unbounded part  $\Omega_2$  and a bounded part  $\Omega_1$  containing the support of  $f$  (see Figure 1). Then (1.1) can alternatively be formulated as follows:

$$\begin{aligned} (1.3a) & \quad \left\{ \begin{array}{l} -\Delta u_1 = f \quad \text{in } \Omega_1, \\ -\Delta u_2 = f = 0 \quad \text{in } \Omega_2, \\ u_1 = u_2 \quad \text{on } \Gamma_2, \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \lambda \quad \text{on } \Gamma_2, \\ u_1 = 0 \quad \text{on } \Gamma, \end{array} \right. \\ (1.3b) & \\ (1.3c) & \\ (1.3d) & \\ (1.3e) & \end{aligned}$$

where  $u_i = u|_{\Omega_i}$ ,  $i = 1, 2$ , and  $\partial/\partial n$  denotes the outward normal derivative to  $\Gamma_2 = \partial\Omega_2$  (see Figure 1). The equations (1.3a) and (1.3b) signify a decomposition into two problems in the separate domains  $\Omega_1$  and  $\Omega_2$ , while (1.3c) and (1.3d) reflect the appropriate *coupling* of these two problems.

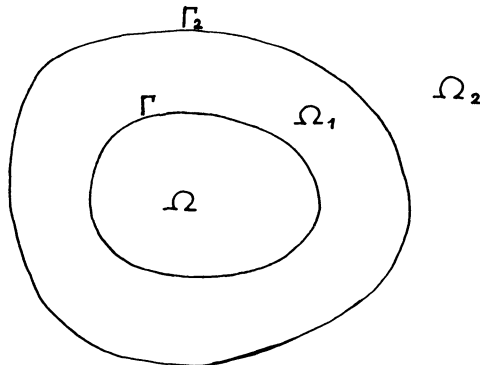


FIGURE 1

**2. A Variational Formulation of the Model Problem.** Let us now give a variational formulation of (1.2). Since  $-\Delta u = f$  in  $\Omega_1$ , we find, using Green's formula, that

$$(2.1) \quad a(u, v) + \langle v, \lambda \rangle = (f, v) \quad \forall v \in W,$$

where

$$\begin{aligned} \lambda &= \frac{\partial u}{\partial n} |_{\Gamma_2}, & a(u, v) &= \int_{\Omega_1} \nabla u \cdot \nabla v \, dx, \\ \langle v, \lambda \rangle &= \int_{\Gamma_2} v \lambda \, ds, & (f, v) &= \int_{\Omega_1} f v \, dx, \\ W &= \{v \in H^1(\Omega_1) : v = 0 \text{ on } \Gamma\}. \end{aligned}$$

Moreover, since  $-\Delta u = 0$  in  $\Omega_2$ , we find, using Green's formula and (1.2), that (cf. [6]),

$$(2.2) \quad \frac{1}{2}u(x) = \int_{\Gamma_2} u(y)G_n(x, y) \, ds_y - \int_{\Gamma_2} \lambda(y)G(x, y) \, ds_y + \alpha, \quad x \in \Gamma_2,$$

$$(2.3) \quad u(x) = \int_{\Gamma_2} u(y)G_n(x, y) \, ds_y - \int_{\Gamma_2} \lambda(y)G(x, y) \, ds_y + \alpha, \quad x \in \Omega_2,$$

where

$$G(x, y) = \frac{1}{2\pi} \log|x - y|, \quad x \neq y,$$

is the Green's function associated with the two-dimensional Laplacian and

$$G_n(x, y) = \frac{\partial}{\partial n_y} G(x, y), \quad x \neq y, y \in \Gamma_2,$$

with  $n_y$  being the outward unit normal to  $\Gamma_2$  at  $y \in \Gamma_2$ . Let us observe that (1.2) together with (2.3) imply that  $\int_{\Gamma_2} \lambda \, ds = 0$ , since otherwise  $u(x)$  would behave like  $c \log|x|$ ,  $c \neq 0$ , as  $|x| \rightarrow \infty$ , thus contradicting (1.2).

Now, formally multiplying (2.2) by the function  $\mu(x)$  satisfying  $\int_{\Gamma_2} \mu \, ds = 0$ , and integrating over  $\Gamma_2$ , we find that

$$(2.4) \quad b(\lambda, \mu) - \frac{1}{2} \langle u, \mu \rangle + \langle G_n u, \mu \rangle = 0,$$

where

$$(2.5) \quad \begin{aligned} b(\lambda, \mu) &= - \int_{\Gamma_2} \int_{\Gamma_2} \lambda(y)\mu(x)G(x, y) \, ds_x \, ds_y, \\ G_n u(x) &= \int_{\Gamma_2} u(y)G_n(x, y) \, ds_y. \end{aligned}$$

We recall (see [6]) that  $b$  is a continuous bilinear form on  $H^{-1/2}(\Gamma_2) \times H^{-1/2}(\Gamma_2)$ . Moreover,  $b$  is  $H$ -elliptic with

$$H = \{\mu \in H^{-1/2}(\Gamma_2) : \langle 1, \mu \rangle = 0\},$$

i.e., there exists a positive constant  $\beta$  such that

$$(2.6) \quad b(\mu, \mu) \geq \beta |\mu|_{-1/2}^2, \quad \mu \in H.$$

Here  $\langle \cdot, \cdot \rangle$  denotes the duality between  $H^{1/2}(\Gamma_2)$  and  $H^{-1/2}(\Gamma_2)$  and we use the notation

$$|\cdot|_s = \|\cdot\|_{H^s(\Gamma_2)}.$$

Recalling (2.1) and (2.4), we are thus led to the following variational formulation of (1.2): Find  $(u, \lambda) \in W \times H$  such that

$$(2.7a) \quad \begin{cases} a(u, v) + \langle v, \lambda \rangle = (f, v) & \forall v \in W, \\ (2.7b) \quad 2b(\lambda, \mu) - \langle u, \mu \rangle + 2\langle G_n u, \mu \rangle = 0 & \forall \mu \in H. \end{cases}$$

Let us now analyze this problem. First, recalling the trace theorem:

$$(2.8) \quad |\gamma v|_{s-1/2} \leq C_s \|v\|_s \quad \forall v \in H^s(\Omega_1),$$

where  $s > 1/2$ ,  $\gamma v = v|_{\Gamma_2}$  and  $\|\cdot\|_s = \|\cdot\|_{H^s(\Omega_1)}$ , it follows that  $\langle \cdot, \cdot \rangle$  is a continuous bilinear form on  $W \times H$ . Further, since  $v = 0$  on  $\Gamma$  if  $v \in W$ , it follows that  $a(\cdot, \cdot)$  is  $W$ -elliptic, i.e., there is a positive constant  $\beta'$  such that

$$(2.9) \quad a(v, v) \geq \beta' \|v\|_1^2 \quad \forall v \in W.$$

Moreover, since

$$(2.10) \quad G_n(x, y) = -\frac{n_y \cdot (x - y)}{|x - y|^2}, \quad x, y \in \Gamma_2,$$

and  $\Gamma$  is smooth, it follows that  $G_n$  is pseudo-homogeneous of degree zero and thus (see [9]) the integral operator  $G_n$  defined by (2.5) is smoothing. More precisely, one has

$$(2.11) \quad |G_n v|_{s+1} \leq C_s |v|_s \quad \forall v \in H^s(\Gamma_2).$$

In order to analyze (2.7), it is convenient to introduce the following simplified problem obtained by omitting the term  $\langle G_n u, \mu \rangle$ : Given  $\hat{g} = (g_1, g_2, g_3)$  find  $(w, \theta) \in W \times H$  such that

$$(2.12a) \quad \begin{cases} a(w, v) + \langle v, \theta \rangle = (g_1, v) + \langle v, g_2 \rangle & \forall v \in W, \\ (2.12b) \quad 2b(\theta, \mu) - \langle w, \mu \rangle = \langle g_3, \mu \rangle & \forall \mu \in H. \end{cases}$$

We shall see that due to (2.11) the full problem (2.7) is a compact perturbation of the simplified problem (2.12).

Let us now formulate (2.7) and (2.12) as operator equations. To this end we introduce the continuous bilinear forms

$$A, B, K: V \times V \rightarrow \mathbf{R},$$

where  $V = W \times H$ , and the corresponding continuous linear mappings

$$A, B, K: V \rightarrow V',$$

defined by

$$\begin{aligned}
 B(\hat{u}, \hat{v}) &\equiv [B\hat{u}, \hat{v}] \equiv a(u, v) + \langle v, \lambda \rangle - \langle u, \mu \rangle + 2b(\lambda, \mu), \\
 K(\hat{u}, \hat{v}) &\equiv [K\hat{u}, \hat{v}] = \langle G_n u, \mu \rangle \quad \forall \hat{u} = (u, \lambda), \hat{v} = (v, \mu) \in V, \\
 A &= B + K,
 \end{aligned}$$

where  $[\cdot, \cdot]$  denotes the duality between  $V$  and  $V'$ , the dual of  $V$ . Then (2.12) can be formulated

$$(2.13) \quad B\hat{u} = \hat{g}, \quad \hat{u} = (u, \lambda),$$

i.e.,

$$(2.14) \quad B(\hat{u}, \hat{v}) = [\hat{g}, \hat{v}] \quad \forall \hat{v} = (v, \lambda) \in V,$$

and (2.7) is equivalent to

$$(2.15) \quad A\hat{u} = \hat{f},$$

i.e.,

$$(2.16) \quad A(\hat{u}, \hat{v}) = [\hat{f}, \hat{v}] \quad \forall \hat{v} \in V,$$

with  $\hat{f} = (f, 0, 0)$ .

Let us note that the bilinear form  $B(\cdot, \cdot)$  is  $V$ -elliptic; by (2.6) and (2.9) we have that

$$\begin{aligned}
 (2.17) \quad B(\hat{v}, \hat{v}) &= a(v, v) + 2b(\mu, \mu) \\
 &\geq \beta' \|v\|_1^2 + 2\beta |\mu|_{-1/2}^2 \geq \beta'' \|\hat{v}\|_V^2 \quad \forall \hat{v} \in V,
 \end{aligned}$$

where  $\beta'' = \min(\beta', 2\beta)$  and  $\|\cdot\|_V$  denotes the norm in  $V$ , i.e.,

$$\|\hat{v}\|_V^2 = (\|v\|_1^2 + |\mu|_{-1/2}^2)^{1/2}.$$

LEMMA 1. *The mapping  $B: V \rightarrow V'$  is an isomorphism. Moreover, for  $k \geq 0$  the mapping*

$$B^{-1}: H^{k-1}(\Omega_1) \times H^{k-1/2}(\Gamma_2) \times H^{k+1/2}(\Gamma_2) \rightarrow H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2),$$

*defined by  $B\hat{u} = \hat{g}$  is continuous.*

*Proof.* The first statement of the lemma follows directly from the  $V$ -ellipticity (2.17). The regularity result is proved in the Appendix below.  $\square$

Let us now return to the original problem  $A\hat{u} = \hat{f}$ . Since  $A = B + K$ , this problem can be written after applying  $B^{-1}$ :

$$(2.18) \quad (I + B^{-1}K)\hat{u} = B^{-1}\hat{f},$$

where  $I: V \rightarrow V$  is the identity mapping. Now, recalling (2.8) and (2.11), it follows that  $K: V \rightarrow \{0\} \times \{0\} \times H^{3/2}(\Gamma_2)$ , is continuous. Therefore, using Lemma 1 with  $k = 1$ , we see that  $B^{-1}K: V \rightarrow H^2(\Omega_1) \times H^{1/2}(\Gamma_2)$  is continuous. Since  $H^2(\Omega_1) \times H^{1/2}(\Gamma_2)$  is compactly embedded in  $V$ , it follows that  $B^{-1}K: V \rightarrow V$  is compact and

hence (2.18) is an equation of the Fredholm second kind. Thus, to prove existence of a solution to (2.18) or the equivalent original problem (2.15), it is sufficient to prove uniqueness. With this observation it is easy to prove

LEMMA 2. *The mapping  $A: V \rightarrow V'$  is an isomorphism. Moreover, for  $k \geq 0$  the mapping*

$$A^{-1}: H^{k-1}(\Omega_1) \times H^{k-1/2}(\Gamma_2) \times H^{k+1/2}(\Gamma_2) \rightarrow H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2)$$

is continuous.

*Proof.* To prove uniqueness of the solution of the equation  $A\hat{u} = \hat{f}$ , let us assume that  $\hat{w} = (w, \theta) \in V$  and  $A\hat{w} = 0$ , i.e.,

$$(2.19a) \quad \begin{cases} a(w, v) + \langle v, \theta \rangle = 0 & \forall v \in W, \end{cases}$$

$$(2.19b) \quad \begin{cases} 2b(\theta, \mu) - \langle w, \mu \rangle + 2\langle G_n w, \mu \rangle = 0 & \forall \mu \in H. \end{cases}$$

From (2.19a) it follows that

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega_1, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \Gamma_2. \end{cases}$$

Let now  $\tilde{w} \in W^1(\Omega_2)$  be the harmonic extension of  $w$  to  $\Omega_2$ , i.e.,

$$(2.20) \quad \begin{cases} -\Delta \tilde{w} = 0 & \text{in } \Omega_2, \\ \tilde{w} = \gamma w & \text{on } \Gamma. \end{cases}$$

Then, by an argument similar to that leading to (2.4), it follows that

$$(2.21) \quad 2b(\tilde{\theta}, \mu) - \langle w, \mu \rangle + 2\langle G_n w, \mu \rangle = 0, \quad \mu \in H,$$

where  $\tilde{\theta} = \partial \tilde{w} / \partial n|_{\Gamma_2} \in H$ . Combining (2.19b) and (2.21) we find that  $b(\tilde{\theta} - \theta, \mu) = 0 \quad \forall \mu \in H$ , and thus (2.6) shows that  $\theta = \tilde{\theta}$ . But this means that if  $w$  is extended to  $\Omega^c$  by putting  $w = \tilde{w}$  in  $\Omega_2$ , then  $\Delta w = 0$  in  $\Omega^c$ ,  $w \in W^1(\Omega^c)$ , and  $w = 0$  on  $\Gamma$  so that  $w \equiv 0$  and the uniqueness follows. Thus, for any  $\hat{g} \in V'$ , the equation  $A\hat{w} = \hat{g}$  has a unique solution and the continuity of  $A^{-1}: V' \rightarrow V$  follows from the closed graph theorem. This proves the first statement of the lemma.

To prove the regularity result, we use induction on  $k$ . Thus assume that the statement holds for  $k = m - 1$ . Let us consider the equation  $A\hat{w} = \hat{g}$ , where  $\hat{g} \in H^{m-1}(\Omega_1) \times H^{m-1/2}(\Gamma_2) \times H^{m+1/2}(\Gamma_2)$ . By the induction hypothesis, we then have  $\hat{w} \in H^m(\Omega_1) \times H^{m-3/2}(\Gamma_2)$  so that by (2.11)  $K\hat{w} \in \{0\} \times \{0\} \times H^{m+1/2}(\Gamma_2)$ . But the equation  $A\hat{w} = \hat{g}$  can be written

$$B\hat{w} = \hat{g} - K\hat{w},$$

and thus by Lemma 1 we conclude that  $\hat{w} \in H^{m+1}(\Omega_1) \times H^{m-1/2}(\Gamma_2) \times H^{m+1/2}(\Gamma_2)$ . Therefore  $A^{-1}$  maps  $H^{m-1}(\Omega_1) \times H^{m-1/2}(\Gamma_2) \times H^{m+1/2}(\Gamma_2)$  into  $H^{m+1}(\Omega_1) \times H^{m-1/2}(\Gamma_2)$  and the continuity of the mapping follows from the closed graph theorem. This completes the induction step and thus the proof of the lemma.  $\square$

We shall also need the corresponding result for the adjoints  $B^*, A^*: V \rightarrow V'$  defined by

$$[A^*\hat{v}, \hat{w}] = [A\hat{w}, \hat{v}],$$

$$[B^*\hat{v}, \hat{w}] = [B\hat{w}, \hat{v}] \quad \forall \hat{v}, \hat{w} \in V.$$

LEMMA 3. *The mappings  $B^*, A^*: V \rightarrow V'$  are isomorphisms and, for  $k \geq 0$ ,  $B^{*-1}, A^{*-1}: H^{k-1}(\Omega_1) \times H^{k-1/2}(\Gamma_2) \times H^{k+1/2}(\Gamma_2) \rightarrow H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2)$  are continuous.*

The proof is parallel to the proofs of Lemmas 1 and 2.

Remark. As pointed out by the referee, if the outer boundary  $\Gamma_2$  is a circle, then one can solve the equation (2.3) in  $\lambda$  explicitly. More precisely, in this case (2.3) takes the form

$$-\frac{1}{\pi} \int_0^{2\pi} \tilde{\lambda}(\theta) \log \left( 2 \left| \sin \frac{\theta - \eta}{2} \right| \right) R d\theta = \gamma(\eta),$$

where  $\tilde{\lambda}(\theta) = \lambda(R \cos \theta, R \sin \theta)$  and  $\gamma$  is determined by  $u$ . This integral equation has the explicit solution

$$\tilde{\lambda}(\theta) = \frac{1}{2\pi R} \int_0^{2\pi} \frac{d\gamma}{d\eta} \cotan \left( \frac{\eta - \theta}{2} \right) d\eta - \frac{1}{2\pi R(\log R^2)} \int_0^{2\pi} \gamma d\eta,$$

which makes it possible to eliminate  $\lambda$  from (2.7) and thus obtain an equation involving only  $u$ . To see if such a procedure is advantageous from a numerical point of view requires further investigation.  $\square$

**3. The Coupled Procedure. Error Estimates.** Let us now consider a finite element method based on the variational formulation (2.7). Let  $W_h \subset W$  and  $H_h \subset H$  be finite-dimensional spaces depending on the positive parameter  $h$  and set  $V_h = W_h \times H_h$ . Let  $A_h(\cdot, \cdot)$  be a bilinear form approximating  $A(\cdot, \cdot)$  and consider the following discrete problem: Find  $\hat{u}_h = (u_h, \lambda_h) \in V_h$  such that

$$(3.1) \quad A_h(\hat{u}_h, \hat{v}) = (f, v) \quad \forall \hat{v} \in V_h.$$

We shall assume that the spaces  $W_h$  and  $H_h$  satisfy the following approximation hypothesis: For any positive  $\epsilon$ , there exists a constant  $C$  such that

$$(3.2a) \quad \inf_{v \in W_h} \|w - v\|_1 \leq Ch^s \|w\|_{s+1+\epsilon}, \quad 0 \leq s \leq k,$$

$$(3.2b) \quad \inf_{\mu \in H_h} |\theta - \mu|_{-1/2} \leq Ch^s |\theta|_{s-1/2}, \quad 0 \leq s \leq k,$$

where  $k$  is a positive integer. This will correspond to using piecewise polynomials of degree  $k$  for  $W_h$  and degree  $k - 1$  for  $H_h$  (cf. Example 1 below). Note that the functions in  $H_h$  may be chosen to be discontinuous while the functions in  $W_h$  will have to be continuous. Furthermore, we shall assume that there is a constant  $C$  such that

$$(3.3) \quad |A(\hat{v}, \hat{w}) - A_h(\hat{v}, \hat{w})| \leq Ch^k \|\hat{v}\|_V \|\hat{w}\|_V \quad \forall \hat{v}, \hat{w} \in V_h.$$

In Example 1 below we shall in detail consider a finite element method satisfying (3.2) and (3.3) with  $k = 1$ .

We shall now prove that for  $h$  small enough the problem (3.1) admits a unique solution  $\hat{u}_h$  and then estimate the error  $\hat{u} - \hat{u}_h$ . The crucial result is then the following:

LEMMA 4. *There is a positive constant  $c$  such that for  $h$  small enough*

$$(3.4) \quad \sup_{\hat{v} \in V_h; \hat{v} \neq 0} \frac{A_h(\hat{w}, \hat{v})}{\|\hat{v}\|_V} \geq c \|\hat{w}\|_V \quad \forall \hat{w} \in V_h.$$

*Proof.* Given  $\hat{w} \in V_h$  there exists by Lemma 3  $\hat{\psi} = (\psi, H) \in V$  such that

$$(3.5) \quad A(\hat{v}, \hat{\psi}) = (\hat{w}, \hat{v})_V, \quad \hat{v} \in V,$$

where  $(\cdot, \cdot)_V$  denotes the scalar product in  $V$ , and

$$(3.6) \quad \|\hat{\psi}\|_V \leq C \|\hat{w}\|_V.$$

In fact,  $\hat{\psi} = A^{*-1}J\hat{w}$ , where  $J: V \rightarrow V'$ , is the canonical mapping defined by  $[J\hat{w}, \hat{v}] = (\hat{w}, \hat{v})_V, \forall \hat{v}, \hat{w} \in V$ . Furthermore, again by Lemma 3, there exists  $\hat{\psi}_h = (\psi_h, H_h) \in V_h$  such that

$$(3.7) \quad B(\hat{v}, \hat{\psi} - \hat{\psi}_h) = 0 \quad \forall \hat{v} \in V_h,$$

and

$$(3.8) \quad \|\hat{\psi}_h\|_V \leq C \|\hat{\psi}\|_V.$$

Now, using (3.7) and (3.5) with  $v = w$ , we find that

$$\begin{aligned} (3.9) \quad A(\hat{w}, \hat{\psi}_h) &= B(\hat{w}, \hat{\psi}_h) + K(\hat{w}, \hat{\psi}_h) \\ &= B(\hat{w}, \hat{\psi}) + K(\hat{w}, \hat{\psi}_h) = A(\hat{w}, \hat{\psi}) + K(\hat{w}, \hat{\psi}_h - \hat{\psi}) \\ &= \|\hat{w}\|_V^2 + \langle G_n \hat{w}, H_h - H \rangle \\ &\geq \|\hat{w}\|_V^2 - |G_n w|_{3/2} |H - H_h|_{-3/2} \\ &\geq \|\hat{w}\|_V^2 - \|\hat{w}\|_V |H - H_h|_{-3/2}, \end{aligned}$$

since by (2.11) and (2.8),

$$|G_n w|_{3/2} \leq C |w|_{1/2} \leq C \|w\|_1 \leq C \|\hat{w}\|_V.$$

In order to estimate  $|H - H_h|_{-3/2}$ , we shall use the usual duality argument: Given  $v \in H^{3/2}(\Gamma_2)$  let  $\hat{\varphi} = B^{-1}\hat{v}$  where  $\hat{v} = (0, 0, v)$ , i.e.,

$$(3.10) \quad B(\hat{\varphi}, \hat{v}) = \langle v, \mu \rangle \quad \forall \hat{v} = (v, \mu) \in V.$$

By Lemma 2 we then have

$$(3.11) \quad \|\hat{\varphi}\|_{H^2(\Omega_1) \times H^{1/2}(\Gamma_2)} \leq C |v|_{3/2}.$$

Hence, taking  $\hat{v} = \hat{\psi} - \hat{\psi}_h$  in (3.10) and using (3.7), we find that for any  $\hat{\varphi}_h \in V_h$ ,

$$\begin{aligned} \langle v, H - H_h \rangle &= B(\hat{\varphi}, \hat{\psi} - \hat{\psi}_h) = B(\hat{\varphi} - \hat{\varphi}_h, \hat{\psi} - \hat{\psi}_h) \\ &\leq C \|\hat{\varphi} - \hat{\varphi}_h\|_V \|\hat{\psi} - \hat{\psi}_h\|_V \leq C \|\hat{\varphi} - \hat{\varphi}_h\|_V \|\hat{w}\|_V, \end{aligned}$$

where the last inequality follows from (3.6) and (3.8). Thus, using (3.2) with  $s = 1 - \epsilon$  together with (3.11), it follows that

$$\langle v, H - H_h \rangle \leq Ch^{1-\epsilon} |v|_{3/2} \|\hat{w}\|_V, \quad v \in H^{3/2}(\Gamma_2),$$

which proves that

$$\|H - H_h\|_{-3/2} \leq Ch^{1-\epsilon} \|\hat{w}\|_V,$$

where  $0 < \epsilon < 1$ . Returning to (3.9) we thus have

$$A(\hat{w}, \hat{\psi}_h) \geq \|\hat{w}\|_V^2 - Ch^{1-\epsilon} \|\hat{w}\|_V^2 = (1 - Ch^{1-\epsilon}) \|\hat{w}\|_V^2.$$

Finally, recalling (3.3), we conclude that

$$A_h(\hat{w}, \hat{\psi}_h) = A(\hat{w}, \hat{\psi}_h) + A_h(\hat{w}, \hat{\psi}_h) - A(\hat{w}, \hat{\psi}_h) \geq (1 - Ch^{1-\epsilon}) \|\hat{w}\|_V^2.$$

Since  $\|\hat{\psi}_h\|_V \leq C \|\hat{w}\|_V$ , this proves that (3.4) holds for  $h$  sufficiently small and the proof is complete.  $\square$

We can now prove

**THEOREM 1.** *For  $h$  sufficiently small the discrete problem (3.1) admits a unique solution  $\hat{u}_h \in V_h$  and we have the following error estimate:*

$$\|\hat{u} - \hat{u}_h\|_V \leq Ch^k \|u\|_{k+1+\epsilon}.$$

*Proof.* Uniqueness, and hence existence, of a solution of (3.1) for  $h$  sufficiently small follows directly from Lemma 4. Furthermore, using (2.16), (3.1), (3.3) and Lemma 4, we see that for any  $\hat{v}_h \in V_h$ ,

$$\begin{aligned} \|\hat{u}_h - \hat{v}_h\|_V &\leq C \sup_{\hat{v} \in V_h} \frac{A_h(\hat{u}_h - \hat{v}_h, \hat{v})}{\|\hat{v}\|_V} \\ &= C \sup_{\hat{v} \in V_h} \frac{A(\hat{u} - \hat{v}_h, \hat{v}) + A(\hat{v}_h, \hat{v}) - A_h(\hat{v}_h, \hat{v})}{\|\hat{v}\|_V} \\ &\leq C \|\hat{u} - \hat{v}_h\|_V + Ch^k \|\hat{v}_h\|_V. \end{aligned}$$

Thus, choosing  $\hat{v}_h$  according to (3.2), we find that

$$\begin{aligned} \|\hat{u} - \hat{u}_h\|_V &\leq \|\hat{u} - \hat{v}_h\|_V + \|\hat{u}_h - \hat{v}_h\|_V \\ &\leq Ch^k (\|u\|_{k+1+\epsilon} + |\lambda|_{k-1/2}). \end{aligned}$$

Finally, by the trace theorem (2.8), we have

$$|\lambda|_{k-1/2} \leq C \|u\|_{k+1},$$

and the proof is complete.  $\square$

Let us now exhibit a natural finite element method satisfying (3.2) and (3.3) with  $k = 1$ .

*Example 1.* Let  $\Gamma_2$  be chosen so that  $\bar{\Omega} \cup \Omega_1$  is convex and let  $\Omega_1^h \subset \Omega_1$  be a polygonal domain approximating  $\Omega_1$  according to Figure 2. Let  $\Gamma^h$  and  $\Gamma_2^h$  be the corresponding polygonal approximations of  $\Gamma$  and  $\Gamma_2$  so that  $\partial\Omega_1^h = \Gamma^h \cup \Gamma_2^h$ . Let  $S_h = \{S\}$  be the sides of  $\Gamma_2^h$ , let  $h$  be the maximal length of the sides  $S \in S_h$ , and define

$$\tilde{H}_h = \{\mu \in L^2(\Gamma_h): \mu|_S \text{ is constant}, S \in S_h, \langle 1, \mu \rangle_h = 0\},$$

where  $\langle v, \mu \rangle_h = \int_{\Gamma^h} v \mu \, ds$ . Further, let  $T_h = \{T\}$  be a regular\* triangulation of  $\Omega_1^h$  with maximal sidelength at most  $h$  and define

$$\tilde{W}_h = \{v \in H^1(\Omega_1^h): v|_T \text{ is linear}, T \in T_h, v = 0 \text{ on } \Gamma^h\}.$$

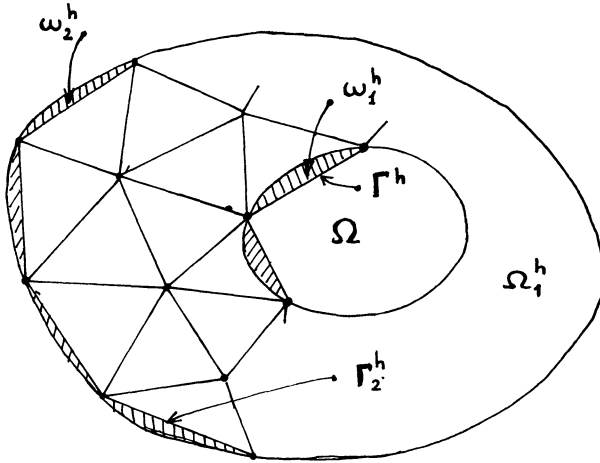


FIGURE 2

In order to formulate a discrete analogue of (2.7), using the spaces  $\tilde{W}_h$  and  $\tilde{H}_h$  and replacing boundary integrals along  $\Gamma$  by integrals along the polygonal boundary  $\Gamma_2^h$ , we have to rewrite the term  $\langle G_n u, \mu \rangle$ . To this end we note that taking  $u \equiv 1$  in (2.2) shows that

$$\int_{\Gamma_2} G_n(x, y) \, ds_y = -\frac{1}{2}, \quad x \in \Gamma_2.$$

Hence recalling (2.10), we have

$$\begin{aligned} \langle G_n u, \mu \rangle &= \int_{\Gamma_2} \int_{\Gamma_2} G_n(x, y) [u(y) - u(x)] \mu(x) \, ds_y \, ds_x - \frac{1}{2} \langle u, \mu \rangle \\ (3.12) \quad &= -\frac{1}{2\pi} \int_{\Gamma_2} \int_{\Gamma_2} \frac{n_y \cdot (x - y)}{|x - y|^2} (u(y) - u(x)) \mu(x) \, ds_y \, ds_x - \frac{1}{2} \langle u, \mu \rangle \\ &\equiv d(u, \mu) - \frac{1}{2} \langle u, \mu \rangle, \end{aligned}$$

\* All angles of the triangles  $T \in T_h$  are bounded below uniformly in  $h$ .

with the obvious definition of  $d(u, \mu)$ . As a discrete analogue of the form  $d(\cdot, \cdot)$ , we now introduce the form  $d_h(\cdot, \cdot)$  defined by

$$\tilde{d}_h(w, \mu) = -\frac{1}{2\pi} \int_{\Gamma_2^h} \int_{\Gamma_2^h} \frac{n_{yh} \cdot (x - y)}{|x - y|^2} (w(y) - w(x))\mu(x) ds_y ds_x,$$

where  $n_{hy}$  is linear on each  $S \in S_h$  and  $n_{hy}$  interpolates the normal  $n_y$  at the vertices of  $\Gamma_2^h$ . We note that, since functions in  $W_h$  are Lipschitz continuous, the form  $\tilde{d}_h(\cdot, \cdot)$  is well defined on  $W_h \times H_h$ .

We can now formulate the following discrete analogue of (2.7): Find  $(\tilde{u}_h, \tilde{\lambda}_h) \in \tilde{W}_h \times \tilde{H}_h$  such that

$$(3.13a) \quad \left\{ \begin{aligned} a_h(\tilde{u}_h, v) + \langle v, \tilde{\lambda}_h \rangle_h &= (f, v)_h \quad \forall v \in \tilde{W}_h, \end{aligned} \right.$$

$$(3.13b) \quad \left\{ \begin{aligned} 2\tilde{b}_h(\tilde{\lambda}_h, \mu) - 2\tilde{u}_h, \mu \rangle_h + 2\tilde{d}_h(\tilde{u}_h, \mu) &= 0 \quad \forall \mu \in \tilde{H}_h, \end{aligned} \right.$$

where

$$a_h(u, v) = \int_{\Omega_1^h} \nabla u \cdot \nabla v dx,$$

$$\langle v, \lambda \rangle_h = \int_{\Gamma_2^h} v \lambda ds,$$

$$\tilde{b}_h(\lambda, \mu) = -\int_{\Gamma_2^h} \int_{\Gamma_2^h} \lambda(y)\mu(x)G(x, y) ds_y ds_x.$$

The problem (3.13) will lead to a nonsymmetric linear system of equations where we have one unknown per node in the triangulation  $T_h$  of  $\Omega_1^h$  and one unknown per side of the polygonal boundary  $\Gamma_2^h$ . The coefficients corresponding to the forms  $a_h(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle_h$  are easy to compute. Algorithms for computing the coefficients corresponding to the forms  $\tilde{b}_h(\cdot, \cdot)$  and  $\tilde{d}_h$  can be found in [2].

Let us now show that the problem (3.13) can be put into the form (3.1) with assumptions (3.2) and (3.3) fulfilled. First, in order to convert the spaces  $\tilde{W}_h$  and  $\tilde{H}_h$  into subspaces  $W_h \subset W$  and  $H_h \subset H$ , we introduce the mapping  $\psi: \Gamma_2^h \rightarrow \Gamma_2$ , where  $\psi(x)$  is the point on  $\Gamma_2$  closest to the point  $x \in \Gamma_2^h$ . For  $h$  small enough  $\psi$  is clearly a bijection. Now, using the mapping  $\psi^{-1}$  to transform integrals along  $\Gamma_2^h$  to integrals along  $\Gamma_2$ , we have

$$\int_{\Gamma_2^h} \mu ds = \int_{\Gamma_2} \mu \circ \psi^{-1} J(\psi^{-1}) ds,$$

where  $J(\psi^{-1}) = |\partial\psi^{-1}/\partial s|$ , and  $\partial/\partial s$  denotes differentiation in the tangential direction to  $\Gamma_2$ . We now define

$$H_h = \{\mu: \mu = J(\psi^{-1})\tilde{\mu} \circ \psi^{-1}, \tilde{\mu} \in \tilde{H}_h\}.$$

Note that if  $\mu = J(\psi^{-1})\tilde{\mu} \circ \psi^{-1} \in H_h$  with  $\tilde{\mu} \in \tilde{H}_h$ , then

$$0 = \int_{\Gamma_2^h} \tilde{\mu} ds = \int_{\Gamma_2} \tilde{\mu} \circ \psi^{-1} J(\psi^{-1}) ds = \int_{\Gamma_2} \mu ds,$$

so that  $H_h \subset H$ .

Furthermore, we extend each function  $\tilde{w} \in \tilde{W}_h$  defined in  $\Omega_1^h$  to a function  $w$  defined in  $\Omega_1$  by setting  $w = 0$  in the “skin”  $\omega_1^h$  with boundary  $\Gamma \cup \Gamma^h$  and finally by setting  $w(y) = w(x)$  for  $y$  on the line segment between  $x \in \Gamma_2^h$  and  $\psi(x)$  thus defining  $w$  in the “skin”  $\omega_2^h$  with boundary  $\Gamma_2 \cup \Gamma_2^h$  (see Figure 2). We denote by  $W_h$  the set of functions obtained in this way.

By changing integrations from  $\Gamma_2^h$  to  $\Gamma_2$  and using the definitions of  $H_h$  and  $W_h$ , the problem (3.13) can now be formulated as follows: Find  $(u_h, \lambda_h) \in W_h \times H_h$  such that

$$(3.14a) \quad \begin{cases} a_h(u_h, v) + \langle v, \lambda_h \rangle = (f, v) & \forall v \in W_h, \\ 2b_h(\lambda_h, \mu) - 2\langle u_h, \mu \rangle + 2d_h(u_h, \mu) = 0 & \forall \mu \in H_h, \end{cases}$$

where

$$b_h(\lambda, \mu) = - \int_{\Gamma_2} \int_{\Gamma_2} \lambda(y)\mu(x)\log|(\psi^{-1}(y) - \psi^{-1}(x))| ds_y ds_x,$$

$$d_h(u, \mu) = -\frac{1}{2\pi} \int_{\Gamma_2} \int_{\Gamma_2} \frac{n_{yh} \cdot (\psi^{-1}(x) - \psi^{-1}(y))}{|\psi^{-1}(x) - \psi^{-1}(y)|^2} (u(y) - u(x))\mu(x) \times J(\psi^{-1}(y)) ds_y ds_x.$$

Let us now check that the assumptions (3.2) and (3.3) are satisfied with  $k = 1$  and

$$A_h(v, w) = a_h(v, w) + \langle w, \lambda \rangle - 2\langle v, \mu \rangle + 2b_h(\lambda, \mu) + 2d_h(v, \mu).$$

To prove (3.2a) let  $w \in H^2(\Omega_1)$  be given and let  $w_h \in \tilde{W}$  interpolate  $w$  at the nodes of  $T_h$ . Then, by well-known interpolation theory (see [1]),

$$\|w - w_h\|_{H^1(\Omega_1^h)} \leq Ch\|w\|_2.$$

By Sobolev’s embedding theorem we have, for any  $\epsilon > 0$ ,

$$\|\nabla w\|_{L^\infty(\Omega_1)} \leq C\|w\|_{2+\epsilon},$$

and hence also

$$\|\nabla w_h\|_{L^\infty(\Omega_1)} \leq C\|w\|_{2+\epsilon}.$$

Since the area of  $\Omega_1 - \Omega_1^h$  is of the order  $\mathcal{O}(h^2)$ , this proves that

$$\|w - w_h\|_{H^1(\Omega_1 \setminus \Omega_1^h)} \leq Ch\|w\|_{2+\epsilon},$$

and thus (3.2a) follows. For a proof of (3.2b) we refer to [6].

It remains to prove (3.3). First, since by the construction of  $W_h$ ,

$$\|w_h\|_{H^1(\omega_2^h)} \leq Ch^{1/2}\|w_h\|_1, \quad w_h \in W_h,$$

we find that

$$(3.15) \quad |a(w_h, v_h) - a_h(w_h, v_h)| = \int_{\omega_2^h} \nabla w_h \cdot \nabla v_h dx \leq Ch\|w_h\|_1 \|v_h\|_1.$$

Next, since (see [5])

$$(3.16) \quad \left| \frac{|\psi^{-1}(x) - \psi^{-1}(y)|}{|x - y|} - 1 \right| \leq Ch^2,$$

it follows easily that

$$(3.17) \quad |b(v, \mu) - b_h(v, \mu)| \leq Ch^2 |v|_0 |\mu|_0.$$

By the “inverse estimate” (see [6])

$$(3.18) \quad |\mu|_0 \leq Ch^{-1/2} |\mu|_{-1/2}, \quad \mu \in H_h,$$

we thus have

$$(3.19) \quad |b(v, \mu) - b_h(v, \mu)| \leq Ch |v|_{-1/2} |\mu|_{-1/2}.$$

Finally, using (3.16) we find that, for  $\epsilon > 0$ ,

$$\begin{aligned} |d(w, \mu) - d_h(w, \mu)| &\leq Ch^2 \int_{\Gamma_h^2} \int_{\Gamma_h^2} \frac{|w(y) - w(x)|}{|x - y|} \mu(x) ds_x ds_y \\ &\leq Ch^2 \left( \int_{\Gamma_2^h} \int_{\Gamma_2^h} \frac{|\mu(x)|^2}{|x - y|^{1-\epsilon}} ds_x ds_y \right)^{1/2} \left( \int_{\Gamma_2^h} \int_{\Gamma_2^h} \frac{|w(y) - w(x)|^2}{|x - y|^{1+\epsilon}} ds_x ds_y \right)^{1/2} \\ &\equiv Ch^2 F_1 F_2, \end{aligned}$$

where we have used Cauchy’s inequality and  $F_1$  and  $F_2$  are defined in the obvious way. Integrating with respect to  $y$  in the factor  $F_1$ , we get  $F_1 \leq |\mu|_0$ . Further (see [5]), for  $0 < \epsilon < 2$ ,  $F_2 \leq C|w|_{\epsilon/2}$ , and therefore, taking  $\epsilon = 1$ , we have again, using (3.18),

$$(3.20) \quad |d(w, \mu) - d_h(w, \mu)| \leq Ch^2 |w|_{1/2} |\mu|_0 \leq Ch^{3/2} \|w\|_1 |\mu|_{-1/2}.$$

Combining (3.15), (3.19), and (3.20), it follows that (3.3) is valid with  $k = 1$  and thus the verification is complete.

We can also construct analogous methods satisfying (3.2) and (3.3) for  $k > 1$  using polynomials of degree  $k$  for  $\tilde{W}_h$  and degree  $k - 1$  for  $\tilde{H}_h$ . In such a case the domain  $\Omega_1$  will be approximated by a domain  $\Omega_1^h$  with piecewise polynomial boundary  $\Gamma^h \cup \Gamma_2^h$  of degree  $k$  approximating  $\Gamma \cup \Gamma_2$ . In the triangulation of  $\Omega_1^h$ , it is then natural to use isoparametric elements of degree  $k$  with one curved edge along  $\Gamma^h \cup \Gamma_2^h$  (cf. [1]). □

**4. Error Estimates in Weaker Norms.** Let us now, using a duality argument, prove an error estimate in a norm weaker than the norm in  $V$ . We shall then make the following assumption: For any  $\epsilon > 0$  there exists a constant  $C$  such that if  $\hat{w}_h \in V_h$  interpolates  $\hat{w} \in X$ , then

$$(4.1) \quad |A(\hat{u}_h, \hat{w}_h) - A_h(\hat{u}_h, \hat{w}_h)| \leq Ch^{k+1-\epsilon} \|\hat{u}\|_X \|\hat{w}\|_X,$$

where  $X = H^2(\Omega_1) \times H^{1/2}(\Gamma_2)$ . We have

**THEOREM 2.** *For any  $\epsilon > 0$  there exists a constant  $C$  such that if  $\hat{u} \in H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2)$ ,  $k \geq 1$ , then*

$$\|\hat{u} - \hat{u}_h\|_{L^2(\Omega_1) \times H^{-3/2}(\Gamma_2)} \leq Ch^{k+1-\epsilon} \|u\|_{k+1+\epsilon}.$$

*Proof.* Given  $\hat{\varphi} \in L^2(\Omega_1) \times H^{3/2}(\Gamma_2)$ , let  $\hat{\psi} \in V$  satisfy

$$(4.2) \quad A(\hat{v}, \hat{\psi}) = [\hat{v}, \hat{\varphi}] \quad \forall \hat{v} \in V.$$

By Lemma 3 we then have

$$\|\hat{\psi}\|_X \leq C \|\hat{\varphi}\|_{L^2(\Omega_1) \times H^{3/2}(\Gamma_2)}.$$

Taking  $\hat{v} = \hat{u} - \hat{u}_h$  in (4.2), recalling (2.16), (3.1), and (3.2), and using (4.1), letting  $\hat{\psi}_h \in V_h$  interpolate  $\hat{\psi}$ , we find that

$$\begin{aligned} [\hat{u} - \hat{u}_h, \hat{\varphi}] &= A(\hat{u}, \hat{u}_h, \hat{\psi}) \\ &= A(\hat{u} - \hat{u}_h, \hat{\psi} - \hat{\psi}_h) + A_h(\hat{u}_h, \hat{\psi}_h) - A(\hat{u}_h, \hat{\psi}_h) \\ &\leq \|\hat{u} - \hat{u}_h\|_V \|\hat{\psi} - \hat{\psi}_h\|_V + Ch^{k+1-\epsilon} \|\hat{u}\|_X \|\hat{\psi}\|_X \\ &\leq C(h^{1-\epsilon} \|\hat{u} - \hat{u}_h\|_V + h^{k+1-\epsilon} \|\hat{u}\|_X) \|\hat{\varphi}\|_{L^2(\Omega_1) \times H^{3/2}(\Gamma_2)}. \end{aligned}$$

Together with Theorem 1, this proves that

$$[\hat{u} - \hat{u}_h, \hat{\varphi}] \leq Ch^{k+1-\epsilon} \|\hat{\varphi}\|_{L^2(\Omega_1) \times H^{3/2}(\Gamma_2)} \quad \forall \hat{\varphi} \in L^2(\Omega_1) \times H^{3/2}(\Gamma_2),$$

and the lemma follows.  $\square$

*Remark.* It is easy to see that the method of Example 1 satisfies (4.1) with  $k = 1$ . Furthermore, if we define  $u_h(x)$  for  $x \in \Omega_2$  by

$$u_h(x) = \frac{1}{2} \int_{\Gamma_2^h} (\tilde{u}_h(y) - \tilde{u}_h(x)) \frac{n_{yh} \cdot (x - y)}{|x - y|^2} ds_y - \frac{1}{2} \int_{\Gamma_2^h} \tilde{\lambda}_h(y) G(x, y) ds_y,$$

then, for all  $x \in \Omega_2$  with  $\text{dist}(x, \Gamma_2) \geq \delta > 0$  and  $h$  sufficiently small, we have  $|u(x) - u_h(x)| \leq C_x h^{2-\epsilon}$ , where the constant  $C_x$  depends on  $\text{dist}(x, \Gamma)$  (cf. [7]).  $\square$

**5. A Symmetrized Procedure.** The solution of the original problem (1.1) can be characterized as the solution of the minimization problem

$$\min_{w \in W^1(\Omega^c)} \left\{ \frac{1}{2} \int_{\Omega^c} |\nabla w|^2 dx - \int_{\Omega^c} f w dx \right\}.$$

Since  $f = 0$  in  $\Omega_2$ , this problem can be formulated in the following way:

$$(5.1) \quad \min_{w \in V} \left\{ \frac{1}{2} \int_{\Omega_1} |\nabla w|^2 dx + \frac{1}{2} \int_{\Omega_2} |\nabla \tilde{w}|^2 dx - \int_{\Omega_1} f w dx \right\},$$

where  $\tilde{w} \in W^1(\Omega_2)$  is the harmonic extension of  $w$  according to (2.20). Since  $\tilde{w}$  is harmonic in  $\Omega_2$ , we have by Green's formula  $\int_{\Omega_2} |\nabla \tilde{w}|^2 = \langle w, Dw \rangle$ , where  $D: H^{1/2}(\Gamma_2) \rightarrow H$  is the continuous operator defined by  $Dw = \partial \tilde{w} / \partial n|_{\Gamma_2}$ . Thus (5.1)

can be formulated as follows:

$$\min_{w \in W} \left\{ \frac{1}{2} \int_{\Omega_1} |\nabla w|^2 dx + \frac{1}{2} \langle w, Dw \rangle - \int_{\Omega_1} fw \right\}.$$

The solution  $u \in W$  of this problem is characterized by the relation

$$(5.2) \quad a(u, v) + \frac{1}{2} \{ \langle v, Du \rangle + \langle u, Dv \rangle \} = (f, v) \quad \forall v \in W.$$

Recalling the formulation (2.7), we have that (2.7b) can equivalently be written  $\lambda = Du$  and thus (2.7c) becomes

$$(5.3) \quad a(u, v) + \langle v, Du \rangle = (f, v) \quad \forall v \in W.$$

By Green's formula we have

$$(5.4) \quad \langle v, Du \rangle = \langle u, Dv \rangle, \quad v, u \in W,$$

and hence (5.2) and (5.3) are equivalent. Thus the problem (5.3) obtained from (2.7), eliminating the variable  $\lambda$ , is in fact *symmetric*. Let us check if the discretized problem (3.13) of Example 1 has the same feature. Introducing the mapping  $D_h: W_h \rightarrow H_h$  defined by

$$b_h(D_h w_h, \mu) - \langle w_h, \mu \rangle + d_h(w_h, \mu) = 0 \quad \forall \mu \in H_h,$$

the problem (3.13) can be written: *Find  $u_h \in W_h$  such that*

$$(5.5) \quad a_h(u_h, v) + \langle v, D_h u_h \rangle = (f, v) \quad \forall v \in W_h.$$

Now, in contrast to (5.4), we have in general  $\langle v, D_h w \rangle \neq \langle w, D_h v \rangle$ , and thus (5.5) will in general lead to a nonsymmetric system of equations.

In order to obtain a symmetric problem, which will facilitate the incorporation of the coupled procedure into existing finite element codes, it is natural to consider the following variant of (5.5): *Find  $u_h \in W_h$  such that*

$$(5.6) \quad a_h(u_h, v) + \frac{1}{2} \{ \langle v, D_h u_h \rangle + \langle u_h, D_h v \rangle \} = (f, v) \quad \forall v \in W_h,$$

or, equivalently, the minimization problem:

$$\min_{w \in W_h} \{ \frac{1}{2} a_h(w, w) + \frac{1}{2} \langle w, D_h w \rangle - (f, w) \}.$$

The problem (5.6) can also be formulated: *Find  $\hat{u}_h \in V_h$  such that*

$$(5.7) \quad \tilde{A}_h(\hat{u}_h, \hat{v}) = (f, v) \quad \forall \hat{v} \in V_h,$$

where

$$\tilde{A}_h(\hat{w}, \hat{v}) = A_h(\hat{w}, \hat{v}) + \frac{1}{2} \{ \langle w, D_h v \rangle - \langle v, D_h w \rangle \}.$$

We shall prove the following lemma which extends the result of Section 3 to the symmetrized problems (5.6), (5.7).

LEMMA 3. *There exists a constant C such that for  $\hat{v}, \hat{w} \in \hat{V}_h$ ,*

$$|A_h(\hat{w}, \hat{v}) - \tilde{A}_h(\hat{w}, \hat{v})| \leq Ch \|\hat{v}\|_V \|\hat{w}\|_V.$$

*Proof.* By the definition of  $D_h v$ , it follows easily that

$$\begin{aligned} \delta_h &\equiv 2|A_h(\hat{w}, \hat{v}) - \tilde{A}_h(\hat{w}, \hat{v})| = |\langle w, D_h v \rangle - \langle v, D_h w \rangle| \\ &= |d_h(w, D_h v) - d_h(v, D_h w)|. \end{aligned}$$

On the other hand, by (5.4) and the definition of  $D$ , we have

$$d(w, Dv) = d(v, Dw), \quad v, w \in W,$$

and thus

$$\begin{aligned} \delta_h &\leq |d_h(w, D_h v) - d(w, D_h v)| + |d_h(v, D_h w) - d(v, D_h w)| \\ &\quad + |d(w, D_h v - Dv) - d(v, D_h w - Dw)| \\ &\equiv \delta_{h1} + \delta_{h2} + \delta_{h3}, \end{aligned}$$

with obvious notation. The first two terms can be estimated using (3.20). Rewriting the remaining term using (3.12), we get

$$\begin{aligned} \delta_{h3} &= |\langle G_n w, D_h v - Dv \rangle - \langle G_n v, D_h w - Dw \rangle| \\ &\leq C[|G_n w|_{3/2} |(D - D_h)v|_{-3/2} + |G_n v|_{3/2} |(D - D_h)v|_{-3/2}]. \end{aligned}$$

Now, by a standard duality argument (see e.g. [6]), we have that

$$|(D - D_h)v|_{-3/2} \leq Ch|v|_{1/2} \leq Ch\|v\|_1.$$

Moreover,

$$|G_n v|_{3/2} \leq C|v|_{1/2} \leq C\|v\|_1,$$

and thus  $\delta_{h3} \leq Ch\|v\|_1 \|w\|_1$ , which completes the proof.  $\square$

*Remark.* The results of Section 4 can also easily be extended to the problem (5.7).  $\square$

**6. Appendix.** We shall here briefly indicate a proof of the regularity result of Lemma 1. We want to prove that, for  $k \geq 0$ ,

$$(6.1) \quad \|w\|_{k+1} + |\theta|_{k-1/2} \leq C(\|g_1\|_{k-1} + \|g_2\|_{k-1/2} + \|g_3\|_{k+1/2})$$

if  $(w, \theta) \in W \times H$  satisfies (2.12). To this end let us first reformulate (2.12b): We have (cf. [4]) that  $\theta \in H$  satisfies (2.12b) if and only if

$$(6.2) \quad \theta = \left[ \frac{\partial \varphi}{\partial n} \right] = \frac{\partial \varphi}{\partial n} \Big|_{\text{int } \Gamma_2} - \frac{\partial \varphi}{\partial n} \Big|_{\text{ext } \Gamma_2},$$

where  $\varphi \in W^1(\mathbf{R}^2)$  satisfies

$$(6.3a) \quad -\Delta \varphi = 0 \quad \text{in } \mathbf{R}^2 \setminus \Gamma_2,$$

$$(6.3b) \quad \varphi = \frac{1}{2}(w + g_3) + c \quad \text{on } \Gamma_2;$$

here  $[\partial\varphi/\partial n]$  denotes the jump in the normal derivative across  $\Gamma_2$  and  $c$  is a suitable constant. By Green's formula we get from (6.2) and (6.3a) that

$$(6.4) \quad D(\varphi, \psi) - \langle \psi, \theta \rangle = 0 \quad \forall \psi \in W^1(\mathbf{R}^2),$$

where

$$D(\varphi, \psi) = \int_{\mathbf{R}^2} \nabla\varphi \cdot \nabla\psi \, dx.$$

Recalling (2.12a), we also have

$$(6.5) \quad a(w, v) + \langle v, \theta \rangle = (g_1, v) + \langle v, g_2 \rangle \quad \forall v \in W.$$

Now, to prove (6.1) for  $k = 0$ , we take  $v = w$  in (6.5),  $\psi = \varphi$  in (6.4) multiply by two and add. Using (6.3b) we then obtain

$$a(w, w) + 2D(\varphi, \varphi) = (g_1, w) + \langle w, g_2 \rangle + \langle g_3, \theta \rangle,$$

and thus (cf. [6])

$$\|w\|_1^2 + \|\varphi\|_{W^1(\mathbf{R}^2)}^2 \leq C[(\|g_1\|_{-1} + \|g_2\|_{-1/2})\|w\|_1 + |\theta|_{-1/2}\|g_3\|_{1/2}].$$

This proves (6.1) in the case  $k = 0$ . For  $k \geq 1$  we parallel the argument in [8] using the  $W \times W^1(\mathbf{R}^2)$ -ellipticity of the form

$$\check{D}(\check{w}, \check{v}) = a(w, v) + 2D(\varphi, \Psi), \quad \check{w} = (w, \varphi), \quad v = (\check{v}, \Psi) \in W \times W_0^1(\mathbf{R}^2). \quad \square$$

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