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THE VALIDITY OF JOHNSON–NÉDÉLEC'S BEM–FEM COUPLING ON POLYGONAL INTERFACES*

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Abstract. In this short article we prove that the classical one-equation (or Johnson–Nédélec) coupling of finite and boundary elements can be applied with a Lipschitz coupling interface. Because of the way it was originally approached from the analytical standpoint, this BEM–FEM scheme required smooth boundaries and hence produced a consistency error in the finite element part. With a variational argument, we prove that this requirement is not needed and that stability holds for all pairs of discrete space, as it inherits the underlying ellipticity of the problem.

Key words. boundary element method–finite element method coupling, Lipschitz domains

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1. Introduction.

Since the first approach to the question [11], from a practical point of view, the coupling of finite and boundary elements has been a very active field of research, providing one of the first working examples of the combined use of two different methods to give the best of each in order to solve problems where both are needed. In this category boundary element method–finite element method (BEM–FEM) coupling is a forerunner of much of what is being done nowadays on multiphysics or multicode computing.

The first mathematical treatment of this coupling appeared in three different papers [3], [2], [6]. It is often referred to as the Johnson–Nédélec coupling (the paper most often quoted is [6]) or one-equation coupling in this last case, as opposed to the two-equation coupling that appeared some time later for reasons that we will explain shortly. A great deal of the analysis in the aforementioned articles hinges on the compactness of the following integral operator, considered as an operator in $H^{1/2}(\Gamma)$:

$$\int_\Gamma \partial_{\nu(\mathbf{y})} \Phi(\,\cdot\,,\mathbf{y})\,\phi(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}) \quad : \quad \Gamma \to \mathbb{R}. \tag{1}$$

Here Γ is a closed Lipschitz curve in the plane (or surface in the space), and Φ is the fundamental solution of the Laplace operator. It is well known in the field of boundary integral equations that (1) is compact when Γ is smooth: Lyapunov regularity, i.e., Hölder continuity of the gradient of any local parametrization, is enough for this. Compactness of this operator fails for polygons/polyhedra. Because of this, the coupling interface for the methods was chosen to be a smooth one, and the finite element grid could not adjust exactly to it. The corresponding difference had to be taken into account as a consistency error term. There is another drawback for the method, based on the fact that (1) is never compact when we are dealing with the linear elasticity system.

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These reasons motivated the appearance, several years later, of the symmetric or two-equation coupling, which was a clear breakthrough both in theory and practice. This new technique originated in separate papers of two different authors [4], [7] and has often been called after its creators, namely, the Costabel–Han coupling method. The symmetry of the method as well as its independence from any compactness property of the operator (1) was obtained at the price of using all four integral operators of the Calderón projector for the Laplacian. In particular, use of the somewhat unpopular hypersingular operator (which is systematically avoided in much of the engineering literature) is needed.

To the best of the author's knowledge, nobody has reported experimental failure of the Johnson–Nédélec coupling when the hypotheses are not met. Although it could be only a question of the need for different analytical techniques, the author's perception is that the main feeling in the mathematical boundary element community was that for Lipschitz interfaces or for the elasticity system, the use of the symmetric coupling technique was a must.

In this short paper, we prove for an extremely simple model problem that Johnson–Nédélec's coupling works for any Lipschitz interface and any choice of discrete spaces. In fact, we will prove that there is not preasymptotic regime: the coupling is stable for any pair of spaces and, as said in the numerical analysis jargon, h is not required to be "sufficiently small."

To do this, we will apply some lateral thinking, using a proof technique that was very recently developed in [9] in the different context of studying the resolvent set for some boundary integral systems associated to the Laplace operator with the overall aim of proving the applicability of convolution-quadrature methods for several scattering problems of transient waves. As the reader will ascertain, the proof uses only simple results but requires some lateral thinking. It is the author's hope that the technique will shed new light in the analysis of BEMs.

Basic knowledge of the theory of boundary integral operators on Lipschitz domains is required for what follows. The main results appear in [5], and the monograph [10] is now considered the standard reference for this kind of result. With the exception of the trace and normal derivative operators, for which we will use traditional symbols, we will use the following character convention: Roman capitals for operators, mathematical (italic) capitals for spaces, Latin small letters for functions on open domains, and Greek letters for functions on the interface Γ.

2. Part I: Statement.

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz boundary Γ, and let $\Omega^e := \mathbb{R}^d \setminus \overline{\Omega}$ be the corresponding exterior domain. The normal vector field on Γ will point outward from Ω, which means that from the point of view of the exterior domain, it will point inward. Consider the jump operators for the trace and the normal derivative

$$[\gamma u] := \gamma u - \gamma^e u, \qquad [\partial_\nu u] := \partial_\nu u - \partial_\nu^e u,$$

where the superscript e makes reference to the exterior value and lack of superscript denotes an interior trace. Let, finally, $f \in L^2(\Omega)$ be given. The model problem we want to study is

$$\left[\begin{array}{rl} -\Delta u + u = f & \text{in } \Omega, \\ [\gamma u] = 0 & \text{on } \Gamma, \\ [\partial_\nu u] = 0 & \text{on } \Gamma, \\ -\Delta u + u = 0 & \text{in } \Omega^e. \end{array}\right. \tag{2}$$

We look for the solution in $H^1(\mathbb{R}^d)$. First of all, some remarks as a vindication for the very simple model equation. We have chosen the Yukawa equation instead of the more traditional Laplace equation to avoid small unimportant annoyances about the existence of energy-free solutions (constant functions are the kernel of the interior Neumann problem), lack of ellipticity of the single-layer operator (only in two dimensions, due to the crucial change of sign of the logarithmic fundamental solution), and conditions at infinity. We are also taking the problem in free space. Insertion of boundary conditions is the usual setting (in that case Ω surrounds another domain where there is no equation) as well as considering variable coefficients in the interior domain. All of these questions are relatively unimportant for the forthcoming analysis, and many of them can be added very easily. In section 4 we will carry out the modifications needed to obtain the same results for the Laplace equation in two and three dimensions.

The first step to deal with (2) consists of using a representation formula for the exterior solution. This is done with help of Green's third formula. To do that, let us introduce some notations. First of all

$$\Phi(r) := \begin{cases} K_0(r)/(2\pi) & \text{when } d = 2, \\ \exp(-r)/(4\pi r) & \text{when } d = 3, \end{cases} \tag{3}$$

is the radial form of the fundamental solution of the Yukawa operator. In (3), K_0 is the modified Bessel function of the second kind (or Macdonald function) and order zero. The single and double layer potentials are defined by understanding the following integral operators as duality products for any $\mathbf{x} \in \mathbb{R}^d \setminus \Gamma$:

$$(\mathrm{S}\lambda)(\mathbf{x}) := \int_\Gamma \Phi(|\mathbf{x}-\mathbf{y}|)\,\lambda(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}),$$

$$(\mathrm{D}\phi)(\mathbf{x}) := \int_\Gamma \partial_{\nu(\mathbf{y})}\left(\Phi(|\mathbf{x}-\mathbf{y}|)\right)\,\phi(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}).$$

The inputs for these potentials are, respectively, $\lambda \in H^{-1/2}(\Gamma)$, $\phi \in H^{1/2}(\Gamma)$. The first step toward the coupling method consists of writing the exterior component of u as

$$u = \mathrm{D}\gamma u - \mathrm{S}\lambda \qquad \text{in } \Omega^e, \tag{4}$$

where $\lambda := \partial_\nu u$. Since transmission of the Cauchy data across Γ is continuous in (3), we can take the interior values to describe the exterior solution. Note that we have already added the symbol λ to represent the normal derivative on Γ, since it is going to be an unknown of the coupled system.

To write the boundary values of (4), two integral operators have to be introduced:

$$\mathrm{V}\lambda := \int_\Gamma \Phi(|\,\cdot\,-\mathbf{y}|)\,\lambda(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}),$$

$$\mathrm{K}\phi := \int_\Gamma \partial_{\nu(\mathbf{y})}\left(\Phi(|\,\cdot\,-\mathbf{y}|)\right)\,\phi(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}).$$

The formulas are the same as for the potentials, but the meaning is completely different. First of all, the outputs of both operators are functions defined on Γ. In fact, the output is in $H^{1/2}(\Gamma)$. Second, we can only understand the integral form as a representation on a dense set of points and, in the case of V, for smooth enough λ. All other cases have to be understood using some duality and/or extension techniques. Full details can be found in the references given in the introduction.

Problem (2) can now be rewritten as follows. The exterior part of the solution is given by (4). For the interior part, we take the classical variational formulation assuming the normal derivative to be known and then we impose an exact match of the traces using the trace value of the representation formula (4). This is tantamount to having the following coupled formulation:

$$\left[\begin{array}{l}(u,\lambda)\in H^1(\Omega)\times H^{-1/2}(\Gamma),\\ \displaystyle\int_\Omega(\nabla u\cdot\nabla v+u\,v)-\langle\lambda,\gamma v\rangle=\int_\Omega f\,v\qquad\forall v\in H^1(\Omega),\\ \left(\tfrac12\mathrm{I}-\mathrm{K}\right)\gamma u+\mathrm{V}\lambda=0.\end{array}\right.\tag{5}$$

We are thus facing a weak problem with a nonlocal essential boundary condition. Angled brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ are used to represent the reciprocal duality of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, understanding always that the first component is in $H^{-1/2}(\Gamma)$ and the second one is in $H^{1/2}(\Gamma)$.

Writing $\mathrm{A}:H^1(\Omega)\to H^1(\Omega)'$ for the operator

$$(\mathrm{A}u)(v):=\int_\Omega(\nabla u\cdot\nabla v+u\,v)\,,$$

it is simple to write (5) in operator form as follows:

$$\mathbb{H}(u,\lambda):=\begin{bmatrix}\mathrm{A} & -\gamma^t\\ \left(\tfrac12\mathrm{I}-\mathrm{K}\right)\gamma & \mathrm{V}\end{bmatrix}\begin{bmatrix}u\\ \lambda\end{bmatrix}=\begin{bmatrix}\ell\\ 0\end{bmatrix},\tag{6}$$

with $\ell(v):=\int_\Omega f\,v$. A quick look at the operator gives an easy idea of how the analysis was approached originally. If K is compact, we can eliminate that part from the operator (it is added back at the end with the help of Fredholm's theory) and then multiply the second equation by two. What is left is

$$\begin{bmatrix}\mathrm{A} & -\gamma^t\\ \gamma & 2\mathrm{V}\end{bmatrix},$$

and it is easy to see that this operator is elliptic: A is elliptic in $H^1(\Omega)$ for obvious reasons, and V satisfies (see [10])

$$\langle\lambda,\mathrm{V}\lambda\rangle\ge C_\Gamma\|\lambda\|^2_{-1/2,\Gamma}\qquad\forall\lambda\in H^{-1/2}(\Gamma).$$

Ellipticity of the principal part of the operator is inherited at the discrete level as stability of Galerkin methods. Stability for the discretization of the full operator equation is obtained asymptotically using standard results of approximation of compact perturbations of operator equations (see [8] for instance). Therefore, we have two important drawbacks: (a) we need K to be compact for the argument to hold; (b) stability is only attained asymptotically when the discrete spaces become closer to the full space. The second one is not recognized as a serious problem in general, but the requirement of compactness of K hinders the possibility of taking Γ polygonal/polyhedral, which would be the best option from the point of view of the finite element discretization.

Let us now go for the discrete equations. Well-posedness of (5) can be found in [6] with no smoothness requirements for Γ but will also follow from our arguments at the discrete level. We take two collections of finite-dimensional subspaces

$$H_h \subset H^1(\Omega), \qquad X_h \subset H^{-1/2}(\Gamma).$$

The discrete equations are then the Galerkin equations for (5):

$$\text{(7)} \qquad \left[\begin{array}{llll} (u_h, \lambda_h) \in H_h \times X_h, & & & \\ \displaystyle\int_\Omega (\nabla u_h \cdot \nabla v_h + u_h\, v_h) & -\langle \lambda_h, \gamma v_h \rangle & = \ell(v_h) & \forall v_h \in H_h, \\ \quad \left\langle \mu_h, \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right) \gamma u_h \right\rangle & +\langle \mu_h, \mathrm{V}\lambda_h \rangle & = 0 & \forall \mu_h \in X_h. \end{array}\right.$$

The bilinear form in (7) induces a discrete operator $\mathbb{H}_h : H_h \times X_h \to H_h' \times X_h'$. Because these discrete equations correspond to a Galerkin method, it is well known that stability is equivalent to a Céa estimate in the natural norm of $H^1(\Omega) \times H^{-1/2}(\Gamma)$:

$$\|u - u_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \le C\Big(\inf_{v_h \in H_h} \|u - v_h\|_{1,\Omega} + \inf_{\mu_h \in X_h} \|\lambda - \mu_h\|_{-1/2,\Gamma} \Big),$$

and both are equivalent to the fact that $\mathbb{H}_h$ has a uniformly bounded inverse (see again [8]). This is in fact the main result of this paper.

THEOREM 1. *The operator $\mathbb{H}_h$ is invertible for any choice of the spaces, and there is a bound for the norm of its inverse independent of the choice of the spaces.*

Let us remark that the proof we will do is not going to use the finite dimensionality of H_h and X_h and will therefore provide at the same time a proof of the well-posedness of the continuous problem.

3. Part II: Proof. We begin with a small modification of the problem. Instead of proving that $\mathbb{H}_h$ has a uniformly bounded inverse, we will do that for the transposed operator $\mathbb{H}_h^t$, which is just the Galerkin discretization of

$$\mathbb{H}^t = \begin{bmatrix} \mathrm{A} & \gamma^t \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right)^t \\ -\gamma & \mathrm{V} \end{bmatrix}.$$

The operator $\mathrm{K}^t : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is formally defined by employing the kernel of K but integrating in the other variable. The reason behind this transposition is the ease of understanding an equation associated to $\mathbb{H}_h^t$ as a nonstandard transmission problem (Lemma 2) for which a variational formulation can be derived (Lemma 3) and proved to be elliptic (Lemma 4). While this might be possible with the direct operator (there is a collection of similar problems in [9] for a series of symmetric problems though), this transposition makes the formulation of the transmission problem especially simple.

Note that one of the jump relations of potentials states that

$$\text{(8)} \qquad \partial_\nu^e \mathrm{S} = -\tfrac{1}{2}\mathrm{I} + \mathrm{K}^t.$$

In fact, if we solve

$$\mathbb{H}^t(u, \psi) = (\ell, 0),$$

what we are doing is proposing a single-layer potential representation for the exterior solution $u = \mathrm{S}\psi$, substituting (8) in the interior variational formulation and imposing equality of trace $\gamma u = \gamma \mathrm{S}\psi = \mathrm{V}\psi$.

We need two more elements before approaching the proof of the Theorem. The canonical inclusion of X_h into $H^{-1/2}(\Gamma)$ will be denoted $\mathrm{P}_h : X_h \to H^{-1/2}(\Gamma)$. Its adjoint $\mathrm{P}_h^t : H^{1/2}(\Gamma) \to X_h'$ consists of restricting an element of $H^{1/2}(\Gamma)$ to be tested only by elements of X_h. The polar set or annihilator of X_h is

$$X_h^\circ := \Big\{ \xi \in H^{1/2}(\Gamma) \,\Big|\, \langle \mu_h, \xi \rangle = 0 \quad \forall \mu_h \in X_h \Big\}.$$

Moreover, because we are in the Hilbert frame and X_h is closed, it follows that given $\mu \in H^{-1/2}(\Gamma)$,

$$\mu \in X_h \qquad \Longleftrightarrow \qquad \langle \mu, \xi \rangle = 0 \quad \forall \xi \in X_h^\circ. \tag{9}$$

Our aim is to show that for any $(d_1, d_2) \in H_h' \times X_h'$, the discrete problem

$$\mathbb{H}_h^t(u_h, \psi_h) = (d_1, d_2) \tag{10}$$

is uniquely solvable and that there exists C independent of the discrete spaces such that

$$\|u_h\|_{1,\Omega} + \|\psi_h\|_{-1/2,\Gamma} \le C \left(\|d_1\|_{H_h'} + \|d_2\|_{X_h'} \right). \tag{11}$$

The proof is going to be given as the consequence of three lemmas.

LEMMA 2. *Let (u_h, ψ_h) solve (10), and let $u^* := \mathrm{S}\psi_h$. Then the pair $(u_h, u^*) \in H_h \times H^1(\mathbb{R}^d)$ satisfies the equations*

$$-\Delta u^* + u^* = 0 \qquad \text{in } \mathbb{R}^d \setminus \Gamma, \tag{12a}$$

$$[\partial_\nu u^*] \in X_h, \tag{12b}$$

$$\mathrm{P}_h^t(\gamma u^* - \gamma u_h) = d_2, \tag{12c}$$

$$\int_\Omega (\nabla u_h \cdot \nabla v_h + u_h\, v_h) - \langle \partial_\nu^e u^*, \gamma v_h \rangle = d_1(v_h) \qquad \forall v_h \in H_h. \tag{12d}$$

Reciprocally, given a solution to (12), the pair $(u_h, [\partial_\nu u^*])$ solves (10).

Proof. It is clear that $u^* = \mathrm{S}\psi_h$ satisfies (12a) and (12b). The coupling condition for the traces (12c) is the second equation of (10). The exterior normal derivative of u^* can be obtained via (8), and therefore (12d) is just the first equation of (10). ∎

The weak formulation of (12) is the content of the following lemma. We will use the bilinear form

$$b\left((u, u^*), (v, v^*)\right) := \int_\Omega (\nabla(u - u^*) \cdot \nabla v + (u - u^*)v) + \int_{\mathbb{R}^d} (\nabla u^* \cdot \nabla v^* + u^*\, v^*),$$

defined for all $(u, u^*), (v, v^*) \in H^1(\Omega) \times H^1(\mathbb{R}^d)$.

LEMMA 3. *Let $\widehat{H}_h := \{(v_h, v^*) \in H_h \times H^1(\mathbb{R}^d) \,|\, \gamma v_h - \gamma v^* \in X_h^\circ\}$. Then $(u_h, u^*) \in H_h \times H^1(\mathbb{R}^d)$ satisfies (12) if and only if*

$$\mathrm{P}_h^t(\gamma u^* - \gamma u_h) = d_2, \tag{13a}$$

$$b\left((u_h, u^*), (v_h, v^*)\right) = d_1(v_h) \qquad \forall (v_h, v^*) \in \widehat{H}_h. \tag{13b}$$

Proof. Let us begin with (13). If we take $v^* \in \mathcal{D}(\mathbb{R}^d \setminus \Gamma)$, the class of $\mathcal{C}^\infty$ functions with compact support not intersecting Γ, then $(0, v^*) \in \widehat{H}_h$, and therefore we can test (13b) with it to obtain that

$$\int_{\mathbb{R}^d} (\nabla u^* \cdot \nabla v^* + u^*\, v^*) = 0 \qquad \forall v^* \in \mathcal{D}(\mathbb{R}^d \setminus \Gamma),$$

which, by the definition of the distributional derivative, is equivalent to the equation $-\Delta u^*+u^*=0$ in $\mathbb{R}^d\setminus\Gamma$. Using Green's formula, we transform the variational equation (13b) into

$$\int_\Omega (\nabla u_h\cdot\nabla v_h+u_h\,v_h)-\langle\partial_\nu u^*,\gamma v_h\rangle+\langle[\partial_\nu u^*],\gamma v^*\rangle=d_1(v_h)\quad\forall(v_h,v^*)\in\widehat{H}_h. \tag{14}$$

We now take an arbitrary $\xi\in X_h^\circ$ and $v^*\in H^1(\mathbb{R}^d)$ such that $\gamma v^*=\xi$. Testing with $(0,v^*)\in\widehat{H}_h$ in (14) we obtain

$$\langle[\partial_\nu u^*],\xi\rangle=0\qquad\forall\xi\in X_h^\circ,$$

and therefore $[\partial_\nu u^*]\in X_h$ by (9). Finally, this and the fact that $\gamma v^*-\gamma v_h\in X_h^\circ$ for any $(v_h,v^*)\in\widehat{H}_h$ mean that we can write

$$\int_\Omega (\nabla u_h\cdot\nabla v_h+u_h\,v_h)+\langle-\partial_\nu u^*+[\partial_\nu u^*],\gamma v_h\rangle=d_1(v_h)\quad\forall(v_h,v^*)\in\widehat{H}_h.$$

Note that v^* does not appear in this formula and that given $v_h\in H_h$, we can find an element $(v_h,v^*)\in\widehat{H}_h$, by simply choosing $v^*\in H^1(\mathbb{R}^d)$ such that $\gamma v^*=\gamma v_h$. Therefore, we can assert the same equality for all $v_h\in H_h$, and (u_h,u^*) solves (12d).

The converse statement is similar. Most of the argument amounts to noticing that if (12b) holds, then

$$-\langle\partial_\nu^e u^*,\gamma v_h\rangle=-\langle\partial_\nu u^*,\gamma v_h\rangle+\langle[\partial_\nu u^*],\gamma v_h\rangle=-\langle\partial_\nu u^*,\gamma v_h\rangle+\langle[\partial_\nu u^*],\gamma v^*\rangle$$

$\forall(v_h,v^*)\in\widehat{H}_h$. Substituting this equality in (12d), we obtain (14). Equation (14) and (12a) prove (13b), which finishes the proof. ◻

LEMMA 4. *Problem* (13) *is uniquely solvable, and there exists* C *independent of the choice of the discrete spaces such that*

$$\|u_h\|_{1,\Omega}+\|u^*\|_{1,\mathbb{R}^d}\le C\left(\|d_1\|_{H_h'}+\|d_2\|_{X_h'}\right). \tag{15}$$

Proof. The bilinear form b in (13) is elliptic because

$$\int_\Omega(|\nabla u|^2+|u|^2)-\int_\Omega(\nabla u\cdot\nabla u^*+u\,u^*)+\int_{\mathbb{R}^d}(|\nabla u^*|^2+|u^*|^2)$$
$$\ge\tfrac12\int_\Omega(|\nabla u|^2+|u|^2)+\tfrac12\int_\Omega(|\nabla u^*|^2+|u^*|^2)+\int_{\Omega^e}(|\nabla u^*|^2+|u^*|^2).$$

Therefore, this property also holds in $\widehat{H}_h$, which shows that there is at most one solution to (13). On the other hand, let us prove the following property: *there exists a linear uniformly bounded operator* $\gamma_h^\dagger:X_h'\to H^1(\mathbb{R}^d)$ *that is a right-inverse of* $\mathrm{P}_h^t\gamma$. Given $d\in X_h'$, we construct $\widetilde{d}\in H^{1/2}(\Gamma)=(H^{-1/2}(\Gamma))'$ by simply imposing

$$\widetilde{d}|_{X_h}=d,\qquad \widetilde{d}|_{X_h^\perp}=0,$$

where $X_h^\perp$ is the orthogonal complement of X_h. This extension operator is linear and has unit norm. We then define $\gamma_h^\dagger d:=\gamma^\dagger\widetilde{d}$, where $\gamma^\dagger$ is any bounded right-inverse of the trace operator.

The pair $(0,\gamma_h^\dagger d_2)$ can be used to make the condition (13a) homogeneous. Since $\mathrm{P}_h^t(\gamma v^* - \gamma v_h) = 0$ is another form of writing $\gamma v^* - \gamma v_h \in X_h^\circ$, in the variable $(u_h, u^*) - (0, \gamma_h^\dagger d_2) \in \widehat{H}_h$, we have a traditional variational problem with no side conditions. Because the bilinear form is bounded and elliptic, we can give a unique solution of it, and the uniform bound (15) is a consequence of this ellipticity property together with the uniform boundedness of the lifting operator. To sketch the proof of this bound, let us consider the product norm

$$\|(u,u^*)\|^2 := \|u\|_{1,\Omega}^2 + \|u^*\|_{1,\mathbb{R}^d}^2,$$

and note that because of the ellipticity property proved above,

$$\begin{aligned}\tfrac{1}{2}\|(u_h, u^* - \gamma_h^\dagger d_2)\|^2 &\le b\Big((u_h, u^* - \gamma_h^\dagger d_2), (u_h, u^* - \gamma_h^\dagger d_2)\Big)\\ &\le \Big(\|d_1\|_{H_h'} + \sqrt{3}\|\gamma_h^\dagger d_2\|\Big)\|(u_h, u^* - \gamma_h^\dagger d_2)\|\end{aligned}$$

(the $\sqrt{3}$ factor appears in an elementary continuity bound for b), which together with the uniform boundedness of $\gamma_h^\dagger$ proves (15). □

With these three lemmas in hand, the proof of the main result is straightforward. Given $(d_1, d_2) \in H_h' \times X_h'$, we can solve (13) or equivalently (12). The solution is uniformly bounded in terms of the right-hand side as in (15). Then Lemma 2 shows that we have a solution to (10) by taking $\psi_h := [\partial_\nu u^*]$. Using (12a), it follows that

$$\|\psi_h\|_{-1/2,\Gamma} = \|[\partial_\nu u^*]\|_{-1/2,\Gamma} \le C\Big(\|\nabla u^*\|_{0,\mathbb{R}^d} + \|\Delta u^*\|_{0,\mathbb{R}^d\setminus\Gamma}\Big) \le \sqrt{2}C\|u^*\|_{1,\mathbb{R}^d}.$$

The constant C depends only on the continuity constant for the trace operator. Then (11) follows from (15). As already mentioned at the beginning of this section, this proves uniform boundedness of the inverse of $\mathbb{H}_h$, i.e., Theorem 1.

The preceding argument works for the continuous operator too. In this case $X_h = H^{-1/2}(\Gamma)$, and therefore $\mathrm{P}_h^t = \mathrm{I}$, $X_h^\circ = \{0\}$, and the condition $[\partial_\nu u^*] \in X_h$ disappears, since it does not impose anything.

Note finally that the key to this argument is the nonstandard transmission boundary value problem (12) and the possibility of obtaining an elliptic variational problem (13) that is equivalent to it. The ellipticity that is hidden in the formulation (6) is thus recovered by moving back to the full domain.

4. Modifications for the Laplace equation. In this section we carry out the needed modifications in the analysis to prove that the stability result holds true when we substitute the Yukawa operator by the Laplace operator. We will need to impose some minor hypotheses on the discrete spaces for reasons that are inherent to the role of constant functions in the Laplace equation. Because of the different behavior of layer potentials in two and three dimensions, we will have to separate the analysis into two subsections.

The problem now is

$$\text{(16)}\qquad \left[\begin{array}{rl} -\Delta u = f & \text{in } \Omega,\\ [\gamma u] = 0 & \text{on } \Gamma,\\ [\partial_\nu u] = 0 & \text{on } \Gamma,\\ -\Delta u = 0 & \text{in } \Omega^e,\\ u = \mathcal{O}(r^{-1}) & \text{at infinity.}\end{array}\right.$$

We will discuss the radiation condition at infinity for the two-dimensional case shortly. Note that in this two-dimensional case, we need that $\int_\Omega f = 0$ (this hypothesis will not come as a surprise for readers acquainted with the differences of electrostatic potentials in the plane and in the space) to ensure the right behavior of u at infinity.

Potentials and integral operators require only the redefinition of the fundamental solution:

$$\Phi(r) := \begin{cases} -1/(2\pi)\,\log|r| & \text{when } d = 2, \\ 1/(4\pi r) & \text{when } d = 3. \end{cases}$$

Note how the fundamental solution does not satisfy the radiation condition in two dimensions, an effect that can induce us to admit logarithmic behavior of solutions at infinity. If we admit unbounded solutions (with at most logarithmic behavior), constant functions enter the kernel of (16), because this problem is set in free space. We will deal with this aspect later on.

The concern of this section is the study of the coupled formulation (compare with (5))

$$\text{(17)} \qquad \begin{bmatrix} (u,\lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma), \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v - \langle \lambda, \gamma v\rangle = \int_\Omega f\, v \qquad \forall v \in H^1(\Omega), \\ \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right)\gamma u + \mathrm{V}\lambda = 0, \end{bmatrix}$$

as well as the stability of Galerkin methods for its discretization. Therefore, we consider spaces H_h and X_h as before and problems

$$\text{(18)} \qquad \begin{bmatrix} (u_h,\lambda_h) \in H_h \times X_h, \\ \displaystyle\int_\Omega \nabla u_h \cdot \nabla v_h - \langle \lambda_h, \gamma v_h\rangle = d_1(v_h) \qquad \forall v_h \in H_h, \\ \left\langle \mu_h, \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right)\gamma u_h\right\rangle + \langle \mu_h, \mathrm{V}\lambda_h\rangle = d_2(\mu_h) \qquad \forall \mu_h \in X_h \end{bmatrix}$$

for arbitrary $d_1 \in H_h'$, $d_2 \in X_h'$.

4.1. Auxiliary results. There are some technicalities needed to adapt the analysis of section 3 to the Laplace equation that require some arguments involving weighted Sobolev spaces. For $d = 2$ and $d = 3$, we consider the spaces

$$W^1(\mathbb{R}^d) := \left\{ u : \mathbb{R}^d \to \mathbb{R} \,\middle|\, \rho\, u \in L^2(\mathbb{R}^d), \quad \nabla u \in (L^2(\mathbb{R}^d))^d \right\},$$

where

$$\rho(\mathbf{x}) := \begin{cases} (1+|\mathbf{x}|^2)^{-1/2}\left(\log(2+|\mathbf{x}|^2)\right)^{-1} & \text{when } d = 2, \\ (1+|\mathbf{x}|^2)^{-1/2} & \text{when } d = 3. \end{cases}$$

These sets are Hilbert spaces when endowed with the natural norm

$$\|u\|^2_{1,\rho,\mathbb{R}^d} := \|\rho\, u\|^2_{1,\mathbb{R}^d} + \|\nabla u\|^2_{1,\mathbb{R}^d}.$$

The following facts are straightforward to prove: (a) constant functions are elements of $W^1(\mathbb{R}^2)$ but not of $W^1(\mathbb{R}^3)$; (b) elements of $W^1(\mathbb{R}^d)$ are locally in H^1 (the weight affects only behavior at infinity); and (c) $H^1(\mathbb{R}^d) \subset W^1(\mathbb{R}^d)$ with continuous injection. Therefore, there is a bounded right-inverse for the trace operator from $W^1(\mathbb{R}^d)$ to $H^{1/2}(\Gamma)$.

An important result concerns the relevance of the gradient seminorm in these spaces. We can state it as follows:

$$\|\nabla u\|_{0,\mathbb{R}^d}^2 + |d-3|\,|\jmath(u)|^2 \geq C\|u\|_{1,\rho,\mathbb{R}^d}^2 \qquad \forall u \in W^1(\mathbb{R}^d). \tag{19}$$

In this formula $\jmath$ (that is needed only when $d = 2$) is any bounded functional such that $\mathbb{P}_0 \cap \ker \jmath = 0$, where $\mathbb{P}_0$ is the space of constant functions. This inequality means that $\|\nabla u\|_{0,\mathbb{R}^d}$ is an equivalent norm to the usual one in $W^1(\mathbb{R}^3)$ and to the quotient norm in $W^1(\mathbb{R}^2)/\mathbb{P}_0$. Formula (19) is an easy consequence of a set of properties for a wide class of weighted Sobolev spaces that can be found in [1] (our space $W^1(\mathbb{R}^d)$ corresponds there to $W^{1,2}_{0,0}(\mathbb{R}^d)$) and relate this definition with equivalent definition as Beppo–Levi spaces.

Solutions of the Laplace equation and single-layer potentials have a very smooth relationship in $\mathbb{R}^3$ and need some minor adjustments in $\mathbb{R}^2$. The set of

$$u \in W^1(\mathbb{R}^d), \qquad \Delta u = 0, \quad \text{in } \mathbb{R}^d \setminus \Gamma$$

can be represented as

$$u = \begin{cases} \mathrm{S}\psi + c, & \psi \in H_0^{-1/2}(\Gamma), \quad c \in \mathbb{R}, \quad \text{when } d = 2, \\ \mathrm{S}\psi, & \psi \in H^{-1/2}(\Gamma) \qquad\qquad\quad \text{when } d = 3, \end{cases} \tag{20}$$

where

$$H_0^{-1/2}(\Gamma) := \{\psi \in H^{-1/2}(\Gamma) \,|\, \langle \psi, 1 \rangle = 0\}.$$

In both cases $\psi = [\partial_\nu u]$. When $d = 2$, c can be recovered as

$$c = \frac{1}{L_\Gamma} \int_\Gamma (\gamma u - \mathrm{V}[\partial_\nu u]), \qquad L_\Gamma := \text{length}(\Gamma) = \langle 1, 1 \rangle.$$

Note that for general $\psi \in H^{-1/2}(\Gamma)$ when $d = 2$

$$\mathrm{S}\psi = -\frac{\langle \psi, 1 \rangle}{2\pi} \int_\Gamma \log|\,\cdot\, - \mathbf{x}| \mathrm{d}\Gamma(\mathbf{x}) + \mathrm{S}\psi_0, \qquad \psi_0 \in H_0^{-1/2}(\Gamma),$$

which shows that considering general densities allows for logarithmically unbounded solutions at infinity, but we still miss constant solutions that are elements of $W^1(\mathbb{R}^2)$ but cannot be represented with potentials.

Finally, for convenience sake, let us write here a simple functional lemma. Its proof follows from elementary compactness arguments and abstract results on self-adjoint operators.

LEMMA 5. *Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a bounded symmetric positive definite bilinear form (i.e., $a(u, u) > 0$ $\forall u \neq 0$). If there exists a compact bilinear form $k : H \times H \to \mathbb{R}$ and $\alpha > 0$ such that*

$$a(u, u) + k(u, u) \geq \alpha \|u\|^2 \qquad \forall u \in H,$$

then there exists $\beta > 0$ such that

$$a(u, u) \geq \beta \|u\|^2 \qquad \forall u \in H.$$

4.2. The three-dimensional case. We have further to assume that

$$\mathbb{P}_0(\Gamma) \subset X_h, \tag{21}$$

that is, constant functions are elements of X_h. The aim is to prove the following result.

THEOREM 6. *For general $d_1 \in H_h'$, $d_2 \in X_h'$, problem* (18) *is uniquely solvable, and the norm of its inverse is bounded independently of the choice of spaces as long as* (21) *holds. The result does not require the finite dimensionality of the spaces, but only their being closed.*

Proof. As with the case of the Yukawa equation, moving to the transposed problem allows for a simple analysis. We can repeat step by step the proofs in section 3, eliminating mass terms $\int u\, v$ from all bilinear forms and substituting the space $H^1(\mathbb{R}^3)$ by $W^1(\mathbb{R}^3)$ in all its occurrences. The key is ellipticity in the proof of Lemma 4. Note first that

$$\begin{aligned} \widehat{H}_h &:= \{(v_h, v^*) \in H_h \times W^1(\mathbb{R}^3) \,|\, \gamma v_h - \gamma v^* \in X_h^\circ\} \\ &\subset \left\{(v, v^*) \in H^1(\Omega) \times W^1(\mathbb{R}^3) \,|\, \int_\Gamma (\gamma v - \gamma v^*) = 0\right\} =: \widehat{H}, \end{aligned}$$

because of (21). Using (19), the compactness of the injection of $H^1(\Omega)$ in $L^2(\Omega)$, and Lemma 5, we can prove that for all $(u, u^*) \in H^1(\Omega) \times W^1(\mathbb{R}^3)$

$$\|\nabla u\|_{0,\Omega}^2 + \|\nabla u^*\|_{0,\mathbb{R}^3}^2 + \left|\int_\Gamma (\gamma u - \gamma u^*)\right|^2 \geq C\Big(\|u\|_{1,\Omega}^2 + \|u^*\|_{1,\rho,\mathbb{R}^3}^2\Big). \tag{22}$$

Therefore, the bilinear form

$$(u, u^*), (v, v^*) \longmapsto \int_\Omega \nabla(u - u^*) \cdot \nabla v + \int_{\mathbb{R}^3} \nabla u^* \cdot \nabla v^*$$

is elliptic in $\widehat{H}$ and hence in $\widehat{H}_h$. ☐

Note that (21) is used to eliminate constants from the interior part of the coupled problem in the ellipticity estimate. It can be easily changed to this other more general hypothesis: there exists a fixed $\phi \in X_h$ (for all h) such that $\langle \phi, 1\rangle \neq 0$.

4.3. The two-dimensional case. Note that in (17),

$$\int_\Omega f = 0 \qquad \Longrightarrow \qquad \lambda \in H_0^{-1/2}(\Gamma),$$

which ensures that the reconstructed exterior solution $u = \mathrm{D}\gamma u - \mathrm{S}\lambda$ has the right behavior at infinity. We now proceed to prove a result similar to Theorem 6 with this new discrete hypothesis:

$$\mathbb{P}_0(\Omega) \subset H_h, \qquad \mathbb{P}_0(\Gamma) \subset X_h. \tag{23}$$

We are going to move constant functions from the interior to the boundary to be able to consider all solutions of the Laplace equation in $W^1(\mathbb{R}^2)$ (recall (20)). This motivates the use of the space

$$H^1_\star(\Omega) := \left\{u \in H^1(\Omega) \,|\, \int_\Omega u = 0\right\}.$$

We also consider $H_{h,\star} := H_h \cap H^1_\star(\Omega)$ and this new problem

$$
(24)\qquad \begin{cases}
(u_{h,\star}, \lambda_h, c_h) \in H_{h,\star} \times X_h \times \mathbb{R}, \\
\displaystyle\int_\Omega \nabla u_{h,\star} \cdot \nabla v_h - \langle \lambda_h, \gamma v_h \rangle = d_1(v_h) \qquad \forall v_h \in H_{h,\star}, \\
\left\langle \mu_h, \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right) \gamma u_{h,\star} \right\rangle + \langle \mu_h, \mathrm{V}\lambda_h \rangle + c_h \langle \mu_h, 1 \rangle = d_2(\mu_h) \qquad \forall \mu_h \in X_h, \\
\langle \lambda_h, 1 \rangle = -d_1(1).
\end{cases}
$$

The relation of the unknowns of (18) and (24) is simple: if (u_h, λ_h) solves (18), then $(u_h - c_h, \lambda_h, c_h)$ solves (24), with $c_h = \frac{1}{|\Omega|}\int_\Omega u_h$. This is a simple consequence of the fact that $(\mathrm{K} - \frac{1}{2}\mathrm{I})1 = 1$. Reciprocally, given a solution to (24), $(u_{h,\star} + c_h, \lambda_h)$ solves (18). Moreover, uniform boundedness of the operator in (18) is equivalent to uniform boundedness of the operator in (24).

As in preceding cases, it will prove useful to work with the transposed problem. For arbitrary $e_1 \in H'_{h,\star}$, $e_2 \in X'_h$, and $e_3 \in \mathbb{R}$, we want to study the problem,

$$
(25)\qquad \begin{cases}
(u_{h,\star}, \varphi_h, c_h) \in H_{h,\star} \times X_h \times \mathbb{R}, \\
\displaystyle\int_\Omega \nabla u_{h,\star} \cdot \nabla v_h + \left\langle \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right)^t \varphi_h, \gamma v_h \right\rangle = e_1(v_h) \qquad \forall v_h \in H_{h,\star}, \\
-\langle \mu_h, \gamma u_{h,\star} \rangle + \langle \mu_h, \mathrm{V}\varphi_h \rangle + c_h \langle \mu_h, 1 \rangle = e_2(\mu_h) \qquad \forall \mu_h \in X_h, \\
\frac{1}{L_\Gamma} \langle \varphi_h, 1 \rangle = e_3.
\end{cases}
$$

The last equation has been scaled for convenience. Using

$$
\psi_h := \varphi_h - e_3 \in X_{h,0} := X_h \cap H_0^{-1/2}(\Gamma)
$$

as unknown, we can concentrate our analysis in solving

$$
(26)\qquad \begin{cases}
(u_{h,\star}, \psi_h, c_h) \in H_{h,\star} \times X_{h,0} \times \mathbb{R}, \\
\displaystyle\int_\Omega \nabla u_h \cdot \nabla v_h + \left\langle \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right)^t \psi_h, \gamma v_h \right\rangle = d_1(v_h) \qquad \forall v_h \in H_{h,\star}, \\
-\langle \mu_h, \gamma u_h \rangle + \langle \mu_h, \mathrm{V}\psi_h \rangle + c_h \langle \mu_h, 1 \rangle = d_2(\mu_h) \qquad \forall \mu_h \in X_h,
\end{cases}
$$

where

$$
d_1(v_h) := e_1(v_h) + \left\langle \left(\tfrac{1}{2}\mathrm{I} - \mathrm{K}\right)^t 1, \gamma v_h \right\rangle e_3, \qquad d_2(\mu_h) := e_2(\mu_h) + \langle 1, \mathrm{V}\psi_h \rangle\, e_3.
$$

Solutions of (26) can be used to define solutions of

$$
(27)\qquad \begin{cases}
(u_{h,\star}, u^*) \in H_{h,\star} \times W^1(\mathbb{R}^2), \\
-\Delta u^* = 0 \qquad \text{in } \Omega^e, \\
[\partial_\nu u^*] \in X_h, \\
\mathrm{P}_h^t(\gamma u^* - \gamma u_{h,\star}) = d_2, \\
\displaystyle\int_\Omega \nabla u_{h,\star} \cdot \nabla v_h - \langle \gamma v_h, \partial_\nu^e u^* \rangle = d_1(v_h) \qquad \forall v_h \in H_{h,\star}
\end{cases}
$$

by taking $u^* = \mathrm{S}\psi_h + c_h$. The process can be applied in the reverse direction as explained in section 4.1. The only needed ingredient to complete our analysis is the proof of

$$\|\nabla u\|_{0,\Omega}^2 + \|\nabla u^*\|_{0,\mathbb{R}^2}^2 + \left|\int_\Gamma (\gamma u - \gamma u^*)\right|^2 \geq C\Big(\|u\|_{1,\Omega}^2 + \|u^*\|_{1,\rho,\mathbb{R}^2}^2\Big)$$

$\forall (u, u^*) \in H^1_\star(\Omega) \times W^1(\mathbb{R}^2)$. This inequality follows from similar arguments as those rendering (22). Note the additional care in elimination of constants from the interior domain. All the remaining steps in the proof follow as in sections 3 and 4.2.

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