Orbital Dynamics: Formulary

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1 Introduction

Newton's law of motion:
The net force on an object is equal to the mass of the object multiplied by its acceleration.

(1) $$\mathbf{F}(t) = m\mathbf{a}(t)$$

where

$\mathbf{F}(t)$:	the net force acting on the object at time t.
m :	the mass of the object.
$\mathbf{x}(t)$:	position of the object at time t, in an inertial frame.
$\mathbf{v}(t) = \dot{\mathbf{x}}(t)$:	velocity of the object at time t.
$\mathbf{a}(t) = \ddot{\mathbf{x}}(t)$:	acceleration of the object at time t.

Newton's law of gravity:
The attractive force F between two bodies is proportional to the product of their masses m_1 and m_2, and inversely proportional to the square of the distance r between them:

(2) $$F = G\frac{m_1 m_2}{r^2} .$$

The constant of proportionality, G, is the gravitational constant.

$$\begin{aligned} G &= (6.67428 \pm 0.00067) \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2} \\ &= (6.67428 \pm 0.00067) \times 10^{-11}\ \mathrm{N}\ \mathrm{m}^2\ \mathrm{kg}^{-2} . \end{aligned}$$

The n–body problem
n point masses m_i with positions $\mathbf{R}_i$ (with respect to an inertial frame) move under the influence of gravity. Let $\mathbf{r}_{ij} := \mathbf{R}_j - \mathbf{R}_i$ be the position of the point mass m_j relative to the position of the point mass m_i. The equations of motion are

(3) $$m_i\ddot{\mathbf{R}}_i = G\sum_{j=1,(j\neq i)}^{n} \frac{m_i m_j}{r_{ij}^2}\mathbf{e}_{ij}, \qquad (i = 1, 2, \ldots, n)$$

where $\mathbf{e}_{ij} := \frac{1}{r_{ij}}\mathbf{r}_{ij}$. Let $\mathbf{R} := \sum_i m_i\mathbf{R}_i / \sum_i m_i$ be the *centre of mass.*

The 10 classical first integrals (conserved quantities)

(4) $$\mathbf{R}(t) = \mathbf{C}_1 t + \mathbf{C}_2$$

(5) $$\sum_{i=1}^{n} m_i(\mathbf{R}_i \times \dot{\mathbf{R}}_i) = \mathbf{C}_3 \qquad \text{(angular momentum)}$$

(6) $$T + V = C_4 \qquad \text{(total energy)}$$

$T := \frac{1}{2}\sum_{i=1}^{n} m_i\dot{R}_i^2$ is the kinetic energy and $V := -G\sum_{i=1}^{n}\sum_{j\neq i}\frac{m_i m_j}{r_{ij}}$ is a potential of the force field $\mathbf{F}$. The components of $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ and C_4 are the 10 classical first integrals.

2 The Two–Body Problem: Orbits

Consider two point masses m_1 and m_2 under the influence of gravity. Let $\mathbf{R} : (m_1 + m_2)\mathbf{R} = m_1\mathbf{R}_1 + m_2\mathbf{R}_2$ be the centre of mass, $\mathbf{r} := \mathbf{R}_2 - \mathbf{R}_1$ be the relative position of m_2 with respect to m_1. Then

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\mathbf{e}_r = -\frac{\mu}{r^3}\mathbf{r} \tag{7}$$

with $\mu := G(m_1 + m_2)$.

The angular momentum

$$\mathbf{h} := \mathbf{r} \times \dot{\mathbf{r}} = \text{ constant} \tag{8}$$

is the *specific angular momentum* (angular momentum per mass). It is a constant of motion. Let $\mathbf{v} = \dot{\mathbf{r}} = v_r\mathbf{e}_r + v_\perp\mathbf{e}_\perp$ then $\mathbf{h} := rv_\perp\mathbf{e}_h$ where $\mathbf{e}_h$ is the unit vector in the direction of $\mathbf{h}$.

The orbit equation

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\vartheta} = \frac{p}{1 + e\cos\vartheta} \tag{9}$$

where ϑ is the *true anomaly* , p is the *semilatus rectum* and e is the *excentricity* of the conic.

$e \in [0, 1)$:	ellipse ($e = 0$: circle)
$e = 1$:	parabola
$e > 1$:	hyperbola

The energy integral
More precisely: the specific energy, energy per mass.
Kinetic energy: $\frac{1}{2}v^2$
Potential energy: $-\frac{\mu}{r}$
According to (6) the (specific) energy

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = \text{constant} \tag{10}$$

is constant along orbits of the two–body problem. Equation (10) relates the velocity to the radius along an orbit. Computing the energy at the periapsis yields

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2p}(1 - e^2) \tag{11}$$

Formulae for the radial and the perpendicular components of the velocity

$$v_r = \frac{h}{p}e\sin\vartheta \tag{12}$$

$$v_\perp = \frac{h}{p}(1 + e\cos\vartheta) \tag{13}$$

Circular orbits ($e = 0$)

(14) $$\mathcal{E} = -\frac{\mu}{2r} \qquad \text{(energy)}$$

(15) $$v = \sqrt{\frac{\mu}{r}} \qquad \text{(velocity)}$$

(16) $$T = \frac{2\pi}{\sqrt{\mu}} r^{3/2} \qquad \text{(period)}$$

Elliptic orbits ($e \in (0, 1)$)
The following formulae hold.

(17) $$r = \frac{a(1 - e^2)}{1 + e\cos\vartheta} \qquad \text{(radius)}$$

(18) $$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \qquad \text{(energy)}$$

Kepler's laws of planetary motion

1. The orbit of every planet is an ellipse with the sun at one of the foci. Thus, Kepler rejected the ancient Aristotelean, Ptolemaic and Copernican belief in circular motion.

2. A line joining a planet and the sun sweeps out equal areas during equal intervals of time as the planet travels along its orbit. This means that the planet travels faster while close to the sun and slows down when it is farther from the sun. With his law, Kepler destroyed the Aristotelean astronomical theory that planets have uniform velocity.

3. The squares of the orbital periods of planets are directly proportional to the cubes of the semi–major axes of their orbits. This means not only that larger orbits have longer periods, but also that the speed of a planet in a larger orbit is lower than in a smaller orbit. More precisely:

   (19) $$T^2 = \frac{4\pi^2}{\mu} a^3$$

   or, for the sun, planet$_1$, planet$_2$ of mass m_s, m_1 and m_2:

   (20) $$\left(\frac{T_1}{T_2}\right)^2 \frac{m_s + m_1}{m_s + m_2} = \left(\frac{a_1}{a_2}\right)^3$$

Parabolic orbits ($e = 1$)

(21) $$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = 0 \qquad \text{(energy)}$$

(22) $$v_{\text{esc}} = \sqrt{\frac{2\mu}{r}} \qquad \text{(escape velocity)}$$

The following holds:
If $v < v_{\text{esc}}$ then the orbit is an ellipse (circles included).

If $v = v_{\mathrm{esc}}$ then the orbit is a parabola.
If $v < v_{\mathrm{esc}}$ then the orbit is a hyperbola.

Hyperbolic orbits $(e > 1)$
The semi–major axis is negative!

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = \frac{\mu}{-2a} > 0 \qquad \text{(energy)} \tag{23}$$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \text{(vis–viva equation)} \tag{24}$$

3 The Two–Body Problem: Position as a Function of Time

3.1 Elliptic orbits

ϑ : True anomaly
E : Excentric anomaly
M : Mean anomaly

The Mean anomaly is the rescaled time; the period T is rescaled to 2π; passage through pericentre at time t_0 corresponds to $M = 0$.

Kepler's equation: (relationship between excentric and mean anomaly)

$$E - e\sin E = M\ . \tag{25}$$

Relationship between true and excentric anomaly

$$\tan\frac{\vartheta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\ . \tag{26}$$

3.2 Hyperbolic orbits

Again, ϑ, E and $M := \frac{h}{ab}(t - t_0)$ are the *true anomaly*, the *excentric anomaly* and the *mean anomaly*.
Kepler's equation for hyperbolic orbits:

$$e\sinh E - E = M\ . \tag{27}$$

Relationship between true and excentric anomaly

$$\tan\frac{\vartheta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{E}{2}\ . \tag{28}$$

3.3 The orbit in space

Consider an inertial system with x, y, z–coordinates. For arbitrary initial conditions $\mathbf{r}(t_0) = \mathbf{r}_0 \neq 0$ and $\dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \neq 0$ there exists a unique solution of (7)

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{29}$$

- The vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not very descriptive. The orbit is easy to describe in a (ξ, η, ζ)–frame with periapsis on the ξ–axis and the ξ, η–plane containing the orbit. Three parameters are needed to describe the position of the (ξ, η, ζ)–frame with respect to the (x, y, z)–frame, for instance the tree Euler angles.
  Ω: longitude of the ascending node ($\angle(\mathbf{e}_x, \mathbf{e}_k)$ where $\mathbf{e}_k = \frac{1}{r_{\text{asc}}}\mathbf{r}_{\text{asc}}$).
  i: inclination ($\angle(\mathbf{e}_z, \mathbf{e}_\zeta) = \angle(\mathbf{e}_z, \mathbf{e}_h)$).
  ω: argument of the pericentre ($\angle(\mathbf{e}_k, \mathbf{e}_p)$).

- To describe the conic section (with periapsis on the positive ξ–axis) two parameters are needed.
  e: describes the shape of the orbit.
  p: describes the size of the orbit.
  Alternatively, e and a could be used.

- To describe a point on the orbit one parameter is needed.
  ϑ: the true anomaly.
  Alternatively, the excentric anomaly E, the mean anomaly M or the elapsed time $t - t_0$ since passage through periapsis may be used.

Orbit elements
The parameters $\Omega, i, \omega, p, e, \vartheta$ are called *elements* of the orbit. They may be determined as follows from given $\mathbf{r}$ and $\mathbf{v}$.

1. Inclination i from

   (30) $$\cos i = \frac{\langle \mathbf{e}_z, h \rangle}{h} = \frac{h_z}{h}$$

   where $\mathbf{h} = \mathbf{r} \times \mathbf{v}$. If $i < \pi/2 = 90^o$ the orbit is prograde, if $i > \pi/2$ the orbit is retrograde.

2. Define $\mathbf{k} := \mathbf{e}_z \times \mathbf{h}$. $\mathbf{k}$ points to the direction of the ascending node. If

   (31) $$\mathbf{h} = \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix}, \qquad \text{then} \quad \mathbf{k} = \begin{pmatrix} -h_y \\ h_x \\ 0 \end{pmatrix}.$$

   $\Omega = \angle(\mathbf{e}_x, \mathbf{k})$ may be determined from

   (32) $$\cos \Omega = \frac{\langle \mathbf{e}_x, \mathbf{k} \rangle}{k} = \frac{-h_y}{\sqrt{h_x^2 + h_y^2}}$$

   If $h_x > 0$, i.e., $k_y > 0$, then $\Omega \in (0, \pi)$.
   If $h_x < 0$, i.e., $k_y < 0$, then $\Omega \in (\pi, 2\pi)$.

3. From the vis–viva equation (24)

   (33) $$a = \frac{r}{2 - \frac{v^2 r}{\mu}}.$$

4. From

$$\mathbf{e} = \left(\frac{v^2}{\mu} - \frac{1}{r}\right)\mathbf{r} - \frac{1}{\mu}\langle \mathbf{r}, \mathbf{v}\rangle \mathbf{v} \; . \tag{34}$$

   determin $e = |\mathbf{e}|$.
   If necessary $p = a(1 - e^2)$.

5. $\omega = \angle(\mathbf{e}, \mathbf{k})$ may be determined from

$$\cos\omega = \frac{\langle \mathbf{e}, \mathbf{k}\rangle}{e\,k} \; . \tag{35}$$

   If $e_3 > 0$ then $\omega \in (0, \pi)$ (e_3 being the third component of $\mathbf{e}$).
   If $e_3 < 0$ then $\omega \in (\pi, 2\pi)$.

6. $\vartheta = \angle(\mathbf{e}, \mathbf{r})$ may be determined from

$$\cos\vartheta = \frac{\langle \mathbf{e}, \mathbf{r}\rangle}{e\,r} \; . \tag{36}$$

   If the distance to the pericentre is
   - increasing, i.e., if $\langle \mathbf{r}, \mathbf{v}\rangle > 0$ then $\vartheta \in (0, \pi)$
   - decreasing, i.e., if $\langle \mathbf{r}, \mathbf{v}\rangle < 0$ then $\vartheta \in (-\pi, 0)$ or $\vartheta \in (\pi, 2\pi)$.

4 Rocket dynamics

There are a lot of different rocket engines, usually categorised as either high– or low–thrust engines. High–thrust engines can provide thrust accelerations significantly larger than the local gravitational acceleration, while for low–thrust engines the thrust acceleration is much smaller than the local gravitational acceleration. To provide thrust, mass is expelled out of the rocket nozzle. Thus the rocket mass is decreasing.

4.1 The thrust

The thrust of a rocket is

$$S = -\dot{m}\,c \tag{37}$$

where $\dot{m}$ describes the loss of mass (it is negative) and where

$$c = v_{\mathrm{e}} + \frac{(p_{\mathrm{e}} - p_{\mathrm{a}})A}{-\dot{m}} \tag{38}$$

is the *effective exhaust velocity*. v_{e} is the velocity of the expelled particles *relative to the rocket*. p_{e} is the pressure of the exhaust at the nozzle exit, p_{a} is the outside ambient pressure (atmospheric pressure, which has value 0 in vacuum), and A is the nozzle exit area.

Assumptions: Rocket in force–free space, $\mathbf{c}$ is constant, one–dimensional motion.

$$m\dot{v} = -\dot{m}\,c \tag{39}$$

or, integrating from t_0 to t_1

$$v_1 - v_0 = c\,\log\frac{m_0}{m_1}, \qquad \frac{m_1}{m_0} = e^{-\frac{\Delta v}{c}} \; . \tag{40}$$

This equation (in either form) is referred to as the *rocket equation*.

4.2 The equations of motion

Let be

γ :	the flight path angle
φ, r :	the polar coordinates
v :	the tangential velocity of the rocket
$a_{\|}$:	the tangential acceleration of the rocket
$a_{\perp}$:	the normal acceleration of the rocket
ρ :	the radius of curvature
$u = \angle(\mathbf{S}, \mathbf{v})$:	controll variable

Then

$$a_{\perp} = -v\,\dot{\gamma} + \frac{v^2}{r}\cos\gamma \tag{41}$$

$$a_{\|} = \dot{v} \tag{42}$$

From these equations one derives the equations of motion of a rocket in a central gravitational field. Thrust S, atmospheric drag $R = \frac{1}{2}\sigma v^2 A c_w$ (σ: density, A: cross sectional area, c_w: drag coefficient).

$$\dot{v} = \frac{S}{m}\cos u - \frac{R}{m} - g\sin\gamma, \qquad (g = \frac{\mu}{r^2}) \tag{43}$$

$$v\,\dot{\gamma} = -(g - \frac{v^2}{r})\cos\gamma + \frac{S}{m}\sin u \tag{44}$$

$$\dot{\varphi} = \frac{v}{r}\cos\gamma \tag{45}$$

$$\dot{r} = v\sin\gamma \tag{46}$$

• There is no lifting force as for aircrafts
• u is a controll variable in order to inject the rocket into the desired orbit. $u = 0$ corresponds to the motion when $\mathbf{S} \parallel \mathbf{v}$.

4.3 Injection into orbit

Velocity in a circular LEO with altitude of 300km: $v_r = 7.728\ldots$ km/s From (43) one gets

$$v = \underbrace{\int_0^\tau \frac{S}{m}dt}_{\text{idealer Antriebsbedarf}} - \underbrace{\int_0^\tau \frac{R}{m}dt}_{\text{Widerstandsverlust}} - \underbrace{\int_0^\tau g\sin\gamma dt}_{\text{gravity loss}} \tag{47}$$

The total loss for injection into a LEO is about 14% ($\approx$ 4% air drag, $\approx$ 10% gravity loss). The required Δv is 9.2 – 9.3 km/s

4.4 Multistage rockets

Let

$$m_0 = m_t + m_s + m_L \tag{48}$$

be the total mass of a rocket at the start. m_t is the mass of propellant, m_s the structural mass and m_L the payload mass. From the *mass ratio* at burnout

$$Z = \frac{m_0}{m_s + m_L} = \frac{m_0}{m_0 - m_t} \tag{49}$$

one immediately gets the characteristic velocity, cf. (40)

$$\Delta v = c \log Z. \tag{50}$$

Moreover, define the *structural coefficient* $\sigma = \frac{m_s}{m_t+m_s}$ and the *payload ratio* $\nu = \frac{m_L}{m_t+m_s} = \frac{m_L}{m_0-m_L}$.

Optimal staging

Let m_{ti}, m_{si}, m_{Li}, respectively, be the propellant mass, the structural mass, the payload mass of the i-th stage, respectively, $i = 1, \ldots, n$. Note that the total mass $m_{0i} = m_{ti} + m_{si} + m_{Li}$ of the i stage is the payload mass of the $(i-1)$-th stage. Define $m_i = m_{ti} + m_{si}$ to be the sum of the propellant mass and the structural mass of the i-th stage.

Problem: For given effective exhaust velocity c_i and stuctural coefficient σ_i of the i-th stage, given payload mass m_L and given $\Delta v_{\text{tot}} := \sum_i \Delta v_i$, $i = 1, \ldots, n$, minimise $M := \sum_{i=1}^{n} m_i$.

Solution: Solve

$$\Delta v_{\text{tot}} - \sum_{i=1}^{n} c_i \log \frac{c_i + 1/\lambda}{c_i \sigma_i} = 0 \tag{51}$$

for λ, then determine

$$Z_i = \frac{c_i + 1/\lambda}{c_i \sigma_i} \tag{52}$$

and

$$\frac{M + m_L}{m_L} = \prod_{i=1}^{n} \frac{(1-\sigma_i) Z_i}{1 - \sigma_i Z_i} \tag{53}$$

5 Orbital Manoeuvres

Part A: Impulsive Orbit Transfer

The limiting case of finite characteristic velocities $\Delta \mathbf{v}$ during a short time $\Delta t \to 0$ is considered. At times t_k, $k = 1, 2, \ldots$ the vehicle undergoes velocity changes $\Delta \mathbf{v}_k = \Delta \mathbf{v}_k^+ - \Delta \mathbf{v}_k^-$

5.1 Hohmann transfer

The Hohman transfer is the minimum–fuel two–impulse transfer between circular orbits. From the vis-viva equation (24) one gets for $0 < r_1 < r_2$

$$\Delta v_{\text{tot}} = \sqrt{\mu} \left[\left(\sqrt{\frac{2}{r_1} - \frac{2}{r_1 + r_2}} - \sqrt{\frac{1}{r_1}} \right) + \left(\sqrt{\frac{1}{r_2}} - \sqrt{\frac{2}{r_2} - \frac{2}{r_1 + r_2}} \right) \right] \tag{54}$$

5.2 Bi–elliptic transfer

For $1 < \alpha < \beta$ the bi–elliptic transfer from a circular orbit of radius r_1 over a point B with $r_B = \beta r_1$ to a circular orbit of radius $r_2 = \alpha r_1$ the total characteristic velocity satisfies

$$\frac{\Delta v_{\mathrm{bi}}}{v_1} = \sqrt{\frac{2}{\beta(\beta+1)}}(\beta - 1) + \sqrt{2\Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)} - \frac{1}{\sqrt{\alpha}} - 1 \tag{55}$$

5.3 Change of the orbit plane

One-impuls changes of the orbit plane of circular orbits are very costly.

$$\frac{\Delta v}{v_{\mathrm{circle}}} = 2\sin(\varphi/2) \tag{56}$$

For large changes of the orbit plane bi–elliptic transfers are more efficient.

5.4 Rendezvous

Synodic period for two objects on coplanar circular orbits

$$S = \frac{2\pi}{n_1 - n_2} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} \tag{57}$$

with angular velocities n_1, n_2.

Initial phase angle for rendezvous with Hohmann transfer:

$$\beta = \pi\Big(1 - \Big(\frac{1 + r_1/r_2}{2}\Big)^{3/2}\Big) \tag{58}$$

Part B: Low Thrust Manoeuvres

The equations of motion are

$$\frac{d}{dt}\mathbf{v} = \frac{d^2}{dt^2}\mathbf{x} = \frac{\mathbf{S}}{m} + \mathbf{g} \tag{59}$$

where $\mathbf{S}$ is the thrust and $\mathbf{g}$ is the gravitational acceleration. For almost circular orbits one gets by integrating

$$\Delta v_{\mathrm{low\ thrust}} = \sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r}} = v_{\mathrm{circle}}(r_0) - v_{\mathrm{circle}}(r) \tag{60}$$

i.e., Δv is equal to the difference of the orbital velocities on the circles.

6 Interplanetary Mission Analysis

6.1 Domain of influence of a planet

Inspect the three–body problem spacecraft–sun–planet. Considering this problem as a perturbed vehicle–planet–two–body problem one gets

$$\ddot{\mathbf{r}}_{pv} = \underbrace{-\frac{G(m_p + m_v)}{r_{pv}^3}\mathbf{r}_{pv}}_{\mathbf{A}_p} \; \underbrace{- G m_s\Big(\frac{\mathbf{r}_{sv}}{r_{sv}^3} - \frac{\mathbf{r}_{sp}}{r_{sp}^3}\Big)}_{\mathbf{S}_s} \tag{61}$$

where $\mathbf{A}_p$ is the acceleration due to the planet and $\mathbf{S}_s$ is the perturbation due to the sun. For short,

$$\ddot{\mathbf{r}}_{pv} - \mathbf{A}_p = \mathbf{S}_s. \tag{62}$$

Analogously,

$$\ddot{\mathbf{r}}_{sv} - \mathbf{A}_s = \mathbf{S}_p. \tag{63}$$

with

$$\mathbf{A}_s = -\frac{G(m_s + m_v)}{r_{sv}^3}\mathbf{r}_{sv}, \qquad \mathbf{S}_p = -Gm_p\Big(\frac{\mathbf{r}_{pv}}{r_{pv}^3} + \frac{\mathbf{r}_{sp}}{r_{sp}^3}\Big) \tag{64}$$

According to Laplace the *domain of influence* of a planet is defined as the set of all points for which

$$\frac{S_p}{A_s} \geq \frac{S_s}{A_p}. \tag{65}$$

The domain of influence of a planet is approximately a ball of radius $(m_p/m_s)^{2/5} r_{sp}$. According to this definition the moon is well inside the domain of influence of the earth.

6.2 Patched conics

Within the domain of influence (sphere of influence) of a planet the two–body problem vehicle-planet is considered. The exit velocity is approximatively equal to $\mathbf{v}_\infty$. Outside of the domain of influence of the planet the two–body problem vehicle-sun is considered. The initial velocity is equal to $\mathbf{v}_{\mathrm{v}} = \mathbf{v}_{\mathrm{planet}} + \mathbf{v}_\infty$

6.3 Flyby or gravity assist

Entrance into the domain of influence with $\mathbf{v}_{-\infty}$, exit with $\mathbf{v}_{+\infty}$. In the sun–vehicle system this leads to $\Delta\mathbf{v} = \mathbf{v}_{\infty} - \mathbf{v}_{-\infty}$. For the magnitude of $\Delta\mathbf{v}$ one has

$$\Delta v = \frac{2v_\infty}{1 + \left(\frac{v_\infty}{v_0}\right)^2 \frac{r_p}{r_0}} \tag{66}$$

where r_0 is the radius of the planet, r_p is the radius of the periapsis and v_0 is the velocity on a circular orbit of radius r_0 (note $r_0 v_0^2 = \mu$).

6.4 The restricted three–body problem

The two primaries with masses m_1, m_2 move on circular orbits around their centre of mass. In a rotating frame with scaled distances the primaries have fixed positions $(-\mu_2, 0, 0)$, $(\mu_1, 0, 0)$ where $\mu_1 = m_2/(m_1 + m_2)$, $\mu_2 = m_1/(m_1 + m_2)$. The equations of motion for a test particle (or a vehicle)

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \tag{67}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \tag{68}$$

$$\ddot{z} = \frac{\partial U}{\partial z} \tag{69}$$

where $U = \frac{1}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$ with ${r_1}^2 = (x+\mu_2)^2 + y^2 + z^2$, ${r_2}^2 = (x-\mu_1)^2 + y^2 + z^2$. More explicitely

$$\ddot{x} - 2\dot{y} - x = -\frac{\mu_1}{r_1^3}(x+\mu_2) - \frac{\mu_2}{r_2^3}(x-\mu_1) \tag{70}$$

$$\ddot{y} + 2\dot{x} - y = -\Big(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\Big)y \tag{71}$$

$$\ddot{z} = -\Big(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\Big)z \tag{72}$$

The Jacoby integral

$$C := x^2 + y^2 + 2\Big(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\Big) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \tag{73}$$

is a constant of motion. There are 5 equilibria: the three Euler points L_1 between the two primaries, L_2 and L_3 on the positive and negative x–axis and the two Lagrange points $L_{4,5} = ((\mu_1 - \mu_2)/2, \pm\sqrt{3}/2)$

7 Perturbations

The Keplerian motion of satellites is perturbed by

- the oblateness of the earth,
- the atmospheric drag,
- the influence of the sun and the moon,
- the radiation pressure, electomagnetic forces, etc.

General assumption: The perturbation is much smaller than gravitation.

7.1 The perturbation equations

The perturbation is given as an acceleration (force/mass). The perturbation

$$\mathbf{F} = F_\xi \mathbf{e}_\xi + F_\eta \mathbf{e}_\eta + F_\zeta \mathbf{e}_\zeta \tag{74}$$

is given in a satellite oriented frame $(\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\zeta)$ with $\mathbf{e}_\xi = (1/r)\mathbf{r}$, $\mathbf{e}_\eta \perp \mathbf{e}_\xi$ in the orbital plane and $\mathbf{e}_\zeta = \mathbf{e}_\xi \times \mathbf{e}_\eta$.

For given $\mathbf{r}(t)$, $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ the osculating elements $a(t), e(t), i(t), \Omega(t), \omega(t), M(t)$ are the elements of the unperturbed Kepler motion corresponding to $\mathbf{r}(t)$ and $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$. For the unperturbed two–body problem the elements are constant, for the perturbed problem the osculating elements vary slowly as time evolves. The osculating elements satisfy the following differential equations.

$$(75) \qquad \dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}\big[F_\xi\, e\sin\vartheta + F_\eta\,(1+e\cos\vartheta)\big]$$

$$(76) \qquad \dot{e} = \sqrt{\frac{a(1-e^2)}{\mu}}\big[F_\xi \sin\vartheta + F_\eta\,(\cos\vartheta + \cos E)\big]$$

$$(77) \qquad \dot{i} = \sqrt{\frac{a(1-e^2)}{\mu}}\,\frac{1}{1+e\cos\vartheta}\,F_\zeta\,\cos(\vartheta+\omega)$$

$$(78) \qquad \dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}}\,\frac{1}{1+e\cos\vartheta}\,F_\zeta\,\frac{\sin(\vartheta+\omega)}{\sin i}$$

$$(79) \qquad \dot{\omega} = \frac{1}{e}\sqrt{\frac{a(1-e^2)}{\mu}}\Big[-F_\xi\cos\vartheta + F_\eta\,\frac{(2+e\cos\vartheta+\cos E)\sin\vartheta}{1+e\cos\vartheta}\Big] - \\ -\dot{\Omega}\,\cos i\,.$$

Introduce the two variables $\nu := \int n dt$ and $\chi := nt_0$ where n is the angular velocity of the mean anomaly M and t_0 is the time of passing through periapsis. The equation for χ is

$$(80) \qquad \dot{\chi} = \frac{\sqrt{\frac{a}{\mu}}(1-e^2)}{e(1+e\cos\vartheta)}\big[F_\xi\,(2e-\cos\vartheta - e\cos^2\vartheta) + F_\eta\,(2+e\cos\vartheta)\sin\vartheta\big]$$

To determine $M = \nu - \chi$ one has to integrate the equation

$$(81) \qquad \dot{\nu} = n = \sqrt{\frac{\mu}{a^3}}.$$

It is often advantageous to take the variable $u := \omega + \vartheta$ a independent variable. A lengthy transformation leads to

$$(82) \qquad \frac{dp}{du} = \frac{2r^3\Gamma}{\mu p}F_\eta$$

$$(83) \qquad \frac{de}{du} = \frac{r^2\Gamma}{\mu e}\big[F_\xi\sin\vartheta + F_\eta\,(\cos\vartheta+\cos E)\big]$$

$$(84) \qquad \frac{di}{du} = \frac{r^3\Gamma}{\mu p}\cos(u)F_\zeta$$

$$(85) \qquad \frac{d\Omega}{du} = \frac{r^3\Gamma}{\mu p}\,\frac{\sin u}{\sin i}F_\zeta$$

$$(86) \qquad \frac{d\omega}{du} = \frac{r^2\Gamma}{\mu e}\Big[-F_\xi\cos\vartheta + F_\eta\,\frac{(2+e\cos\vartheta+\cos E)\sin\vartheta}{1+e\cos\vartheta} - \\ - F_\zeta\,\frac{e}{1+e\cos\vartheta}\,\frac{\sin u}{\tan i}\Big]$$

where

$$(87) \qquad \Gamma = \frac{1}{1 - \frac{r^3}{\mu p}\,\frac{\sin u}{\tan i}\,F_\zeta}\,.$$

7.2 The method of averaging

Consider the differential equation

$$\dot{x} = \varepsilon f(t, x) \tag{88}$$

where ε is a small parameter and where f is T–periodic with respect to t. Then the solutions of the averaged equation

$$\dot{y} = \varepsilon \overline{f}(y) \tag{89}$$

with $\overline{f}(y) := (1/T)\int_0^T f(t, y)dt$ satisfy

$$|x(t) - y(t)| \leq C\varepsilon \qquad \text{for } t \in [0, L/\varepsilon] \ . \tag{90}$$

7.3 Oblateness of the earth

The gravitational potential of the earth is approximated by

$$U = U_0 + U_{J_2} = \frac{\mu}{r} - \frac{\mu r_0^2}{r^3} J_2 (3\sin^2\varphi - 1)/2 \tag{91}$$

where U_{J_2} describes the influence of the oblateness of the earth. The perturbation $\mathbf{F} =$ grad U_{J_2} is

$$\mathbf{F} = -\frac{3\mu r_0^2 J_2}{r^4}\Big[\frac{1}{2}(1 - 3\sin^3 i \sin^2 u)\,\mathbf{e}_\xi + \sin^2 i \sin u \cos u\,\mathbf{e}_\eta + \sin i \cos i \sin u\,\mathbf{e}_\zeta\Big] \tag{92}$$

Setting $\Gamma = 1$ in (82)–(86) one gets after rescaling to the variable t

$$\dot{\overline{\Omega}} = -\frac{3}{2}\Big(\frac{r_0}{a}\Big)^{3.5}\sqrt{\frac{\mu}{r_0^3}}J_2\frac{\cos i}{(1-e^2)^2} = -9.964\Big(\frac{r_0}{a}\Big)^{3.5}\frac{\cos i}{(1-e^2)^2}\ {}^\circ/24\text{h} \tag{93}$$

$$\dot{\overline{\omega}} = \frac{3}{4}\Big(\frac{r_0}{a}\Big)^{3.5}\sqrt{\frac{\mu}{r_0^3}}J_2\frac{5\cos^2 i - 1}{(1-e^2)^2} = 4.982\Big(\frac{r_0}{a}\Big)^{3.5}\frac{5\cos^2 i - 1}{(1-e^2)^2}\ {}^\circ/24\text{h} \tag{94}$$

$$\dot{\overline{a}} = 0 \tag{95}$$

$$\dot{\overline{i}} = 0 \tag{96}$$

$$\dot{\overline{e}} = 0 \tag{97}$$

If $i < 90°$ then $\dot{\overline{\Omega}} < 0$, i.e., the node line drifts westward. If $i > 90°$ then $\dot{\overline{\Omega}} > 0$, i.e., the node line advances eastward.

If $i < 63.4°$ or $i > 116.6°$ then $\dot{\overline{\omega}} > 0$, meaning that the perigee advances in the direction of the satellite. If $63.4° < i < 116.6°$ then the perigee regresses, it moves oposite to the direction of motion.

7.4 Atmospheric drag

For nearly circular LEO–orbits one has approximately

$$F_\xi = 0, \quad F_\eta = -\frac{1}{2}\rho c_w A v^2/m, \quad F_\zeta = 0. \tag{98}$$

From (75) one gets with $a = r$ and $v^2 = \mu/r$

$$\dot{r} = -\sqrt{\mu r}\rho c_w A/m < 0. \tag{99}$$

From (84)–(86) one gets $\dot{\bar{i}} = 0$, $\dot{\bar{\Omega}} = 0$, $\dot{\bar{\omega}} = 0$, meaning that the orbit plane and the direction of the perigee remain constant.

For noncircular orbits one gets

$$\begin{aligned}
\dot{\bar{p}} &< 0 & (100)\\
\dot{\bar{e}} &< 0 & (101)\\
\dot{\bar{\Omega}} &= 0 & (102)\\
\dot{\bar{i}} &= 0 & (103)\\
\dot{\bar{\omega}} &= 0 & (104)
\end{aligned}$$

The orbit becomes smaller and closer to a circular orbit. The orbit plane and the direction of the perigee remain constant.

8 Attitude dynamics

An Example: the dumbbell satellite

Two masses m are connected with a massless rod of length l. Positions of the masses:

$$\begin{aligned}
x_{1,2} &= r\cos\varphi \pm l\cos(\varphi + \vartheta) & (105)\\
y_{1,2} &= r\sin\varphi \pm l\sin(\varphi + \vartheta) & (106)
\end{aligned}$$

With the kinetic and the potential energy

$$T = m[\dot{r}^2 + r^2\dot{\varphi}^2 + l^2(\dot{\varphi} + \dot{\vartheta})^2] \tag{107}$$

$$U = -m\left(\frac{\mu}{r_1} + \frac{\mu}{r_2}\right) \tag{108}$$

and the Lagrange function $L = T - U$ one derives the equations of motion from the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{109}$$

for $q = r, \varphi, \vartheta$. Taking the limit $l \to 0$ one gets

$$\begin{aligned}
\ddot{r} - r\dot{\varphi}^2 &= -\frac{\mu}{r^2} & (110)\\
\frac{d}{dt}(r^2\dot{\varphi}) &= 0 & (111)\\
\ddot{\vartheta} + \frac{3\mu}{2}\frac{\sin(2\vartheta)}{r^3} &= -\ddot{\varphi} & (112)
\end{aligned}$$

Equations (110) and (111) are the equations for the Kepler problem in polar coordinates. They are decoupled from (112). Equation (112) describes the attitude dynamics of the dumbbell satellite.

For circular orbits (112) degenerates to the pendulum equation

$$\ddot{\vartheta} + \frac{3\mu}{2r_0^3}\,\sin(2\vartheta) \;=\; 0\,. \tag{113}$$

The radial equilibrium solution $\vartheta = 0$ is stable, the tangential equilibrium solution $\vartheta = \pi/2$ is unstable.

To investigate (112) it is convenient to replace the time t by the excentric anomaly E. One gets

$$\vartheta'' - \vartheta' \frac{e \sin E}{1 - e\cos\vartheta} + \frac{3}{2}\,\frac{\sin(2\vartheta)}{1 - e\cos\vartheta} \;=\; \frac{2e\sqrt{1-e^2}\sin E}{(1 - e\cos\vartheta)^2} \tag{114}$$

where $'$ denotes the derivative d/dE with respect to E.

Appendix

Vector identities

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{a}, \mathbf{b} \rangle \mathbf{c} \\ \langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle &= \langle \mathbf{c}, \mathbf{a} \times \mathbf{b} \rangle \end{aligned}$$

Astronomical constants

The Sun

$$\begin{aligned} \text{mass} &= 1.989 \cdot 10^{30}\,\text{kg} \\ \text{radius} &= 6.9599 \cdot 10^{5}\,\text{km} \\ \mu_{\text{sun}} = G\,m_{\text{sun}} &= 1.327 \cdot 10^{11}\,\text{km}^3/\text{s}^2 \end{aligned}$$

The Earth

$$\begin{aligned} \text{mass} &= 5.974 \cdot 10^{24}\,\text{kg} \\ \text{radius} &= 6.37812 \cdot 10^{3}\,\text{km} \\ \mu_{\text{earth}} = G\,m_{\text{earth}} &= 3.986 \cdot 10^{5}\,\text{km}^3/\text{s}^2 \\ \text{mean distance from sun} = 1\text{ au} &= 1.495978 \cdot 10^{8}\,\text{km} \end{aligned}$$

The Moon

$$\begin{aligned} \text{mass} &= 7.3483 \cdot 10^{22}\,\text{kg} \\ \text{radius} &= 1.738 \cdot 10^{3}\,\text{km} \\ \mu_{\text{moon}} = G\,m_{\text{moon}} &= 4.903 \cdot 10^{3}\,\text{km}^3/\text{s}^2 \\ \text{mean distance from earth} &= 3.844 \cdot 10^{5}\,\text{km} \\ \text{orbit eccentricity} &= 0.0549 \\ \text{orbit inclination (to ecliptic)} &= 5^\circ\,09' \end{aligned}$$

Physical characteristics of the planets

Planet	Equatorial radius (units of R_{earth})	Mass (units of M_{earth})	Siderial rotation period	Inclination of equator to orbit plane
Mercury	0.382	0.0553	58d 16h	$\approx 2^\circ$
Venus	0.949	0.8149	243d(retro)	$177^\circ\,18'$
Earth	$1,000$	1.000	23h 56m 04s	$23^\circ\,27'$
Mars	0.532	0.1074	24h 37m 23s	$25^\circ\,11'$
Jupiter	11.209	317.938	9h 50m	$3^\circ\,07'$
Saturn	9.49	95.181	10h 14m	$26^\circ\,44'$
Uranus	4.007	14.531	17h 54m	$97^\circ\,52'$
Neptune	3.83	17.135	19h 12m	$29^\circ\,36'$
Pluto	0.18	0.0022	6d 9h 18m	$122^\circ\,46'$

Elements of the planetary orbits

Planet	semimajor axis (in au)	eccentricity	siderial period	inclination to ecliptic plane
Mercury	0.3871	0.2056	87.969d	$\approx 7^\circ\,00'$
Venus	0.7233	0.0068	224.701d	$3^\circ\,24'$
Earth	1.0000	0.0167	365.256d	$0^\circ\,00'$
Mars	1.5237	0.0934	1y 321.73d	$1^\circ\,51'$
Jupiter	5.2028	0.0483	11y 314.84d	$1^\circ\,19'$
Saturn	9.5388	0.0560	29y 167d	$2^\circ\,30'$
Uranus	19.1914	0.0461	84y 7.4d	$0^\circ\,46'$
Neptune	30.0611	0.0097	164y 280.3d	$1^\circ\,47'$
Pluto	39.5294	0.2482	247y 249d	$17^\circ\,09'$