

Pro memoria

$$E_k(z) = 1 + C_k \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m, \quad q = e^{2\pi iz}$$

with

k	2	4	6	8	10	12
$C_k$	-24	240	-504	480	-264	$\frac{65520}{691}$

### Exercise 1

a.) It is clear that  $E_4^2 \in \mathbb{M}_8$  and  $E_4 E_6 \in \mathbb{M}_{10}$ .

Since  $\dim \mathbb{M}_8 = \dim \mathbb{M}_{10} = 1$ , we get  $E_4^2 = cE_8$  and  $E_4 E_6 = dE_{10}$ . Comparing the coefficient of  $q^0$  we see  $c = d = 1$

b.) We have

$$\begin{aligned} E_8 &= E_4^2 = \left( 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m \right)^2 = 1 + 480 \sum_{m=1}^{\infty} \sigma_3(m) q^m + \left( 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m \right)^2 \\ &= 1 + 480 \sum_{m=1}^{\infty} \sigma_3(m) q^m + 240^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \sigma_3(n) \sigma_3(m-n) q^m \end{aligned}$$

Comparing the coefficient of  $q^m$ ,  $m \geq 1$  we get

$$480\sigma_7(m) = 480\sigma_3(m) + 240^2 \sum_{n=1}^{m-1} \sigma_3(n) \sigma_3(m-n)$$

Finally we divide by 480 and get

$$\sigma_7(m) = \sigma_3(m) + 120 \sum_{n=1}^{m-1} \sigma_3(n) \sigma_3(m-n)$$

Analog

$$E_{10} = E_4 E_6 = 1 + \sum_{m=1}^{\infty} (240\sigma_3(m) - 504\sigma_5(m)) q^m - 240 \cdot 504 \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \sigma_3(n) \sigma_5(m-n) q^m$$

and

$$11\sigma_9(m) = 21\sigma_5(m) - 10\sigma_3(m) + 5040 \sum_{n=1}^{m-1} \sigma_3(n) \sigma_5(m-n)$$

### Exercise 2

Let  $k \geq 2$ . It is clear that  $g(z)$  is holomorphic in  $\mathbb{H}$  and periodic with Period 1. Watching at the Fourier expansion of each factor, one also sees immediately that  $g(z)$  is holomorphic at infinity. So we only have to show  $g(-1/z) = z^{k+2}g(z)$ . We have

$$E_2(-1/z) = z^2 E_2(z) + \frac{12z}{2\pi i}$$

We differentiate  $f(-1/z) = z^k f(z)$  and multiply by  $z^2$  and get

$$f'(-1/z) = kz^{k+1}f(z) + z^{k+2}f'(z)$$

$$\begin{aligned} g(-1/z) &= \frac{1}{2\pi i} f'(-1/z) - \frac{k}{12} E_2(-1/z) f(-1/z) \\ &= \frac{1}{2\pi i} (kz^{k+1}f(z) + z^{k+2}f'(z)) - \frac{k}{12} \left( z^2 E_2(z) + \frac{12z}{2\pi i} \right) z^k f(z) \\ &= z^{k+2} \left( \frac{1}{2\pi i} f'(z) - \frac{k}{12} E_2(z) f(z) \right) + \left( \frac{1}{2\pi i} kz^{k+1}f(z) - \frac{k}{12} \frac{12z}{2\pi i} z^k f(z) \right) \\ &= z^{k+2} g(z) \end{aligned}$$

This implies  $g(z) \in \mathbb{M}_{k+2}$ .

Now will calculate the Fourier coefficient of  $q^0$  of  $g(z)$ . We have  $f(z) = a_0 + a_1q + O(q^2)$  and  $E_2(z) = 1 - 24q + O(q^2)$ . Notice  $(q^m)' = 2\pi imq^m$

$$g(z) = a_1q + O(q^2) - \frac{k}{12}(1 - 24q + O(q^2))(a_0 + a_1q + O(q^2)) = \frac{-k}{12}a_0 + O(q)$$

We get immediately  $g(z)$  is a cusp form if and only if  $f(z)$  is a cusp form.

### Exercise 3

a.) Using Exercise 2 with  $f = -3E_4$  we see  $E_4E_2 - \frac{3}{2\pi i}E_4'$  is a modular form with coefficient of  $q^0$  equal to 1. Since  $\dim \mathbb{M}_6 = 1$  we get  $E_6 = E_4E_2 - \frac{3}{2\pi i}E_4'$

b.) We get

$$\begin{aligned} E_6 &= E_4E_2 - \frac{3}{2\pi i}E_4' \\ &= \left( 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m \right) \left( 1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)q^m \right) - \frac{3}{2\pi i} \left( -24 \sum_{m=1}^{\infty} \sigma_1(m)(2\pi i)mq^m \right) \\ &= 1 + \sum_{m=1}^{\infty} (240\sigma_3(m) + 24(3m-1)\sigma_1(m))q^m - 24 \cdot 240 \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \sigma_1(n)\sigma_3(m-n)q^m \end{aligned}$$

It follows for  $m \geq 1$

$$-504\sigma_5(m) = 240\sigma_3(m) + 24(3m-1)\sigma_1(m) - 24 \cdot 240 \sum_{n=1}^{m-1} \sigma_1(n)\sigma_3(m-n)$$

Dividing by  $-24$

$$21\sigma_5(m) = (1-3m)\sigma_1(m) - 10\sigma_3(m) + 240 \sum_{n=1}^{m-1} \sigma_1(n)\sigma_3(m-n)$$

Exercise 4

a.) It is clear  $E_{12} - E_6^2 \in \mathbb{M}_{12}$ . Now we take a look at the Fourier expansion:

$$E_{12} - E_6^2 = 1 + \frac{65520}{691}q + O(q^2) - (1 - 504q + O(q^2))^2 = \left( \frac{65520}{691} + 1008 \right) q + O(q^2)$$

Since  $a_0 = 0$  we see that  $E_{12} - E_6^2$  is a cusp form. But  $\dim \mathbb{S}_{12} = 1$ .

So we get  $E_{12} - E_6^2 = c\Delta$ . On the other Hand  $\Delta = (2\pi)^{12} \sum_{m=1}^{\infty} \tau(m)$  with  $\tau(1) = 1$  and  $\tau_m \in \mathbb{Z} \forall m$ . So  $c = (2\pi)^{-12} \frac{762048}{691}$

b.) For simplicity we write  $E_6^2(z) = 1 + \sum_{m=1}^{\infty} b_m q^m$  with  $b_m \in \mathbb{Z}$ . Therefore

$$\frac{762048}{691} \sum_{m=1}^{\infty} \tau(m) = \frac{65520}{691} \sum_{m=1}^{\infty} \sigma_{11}(m) q^m - \sum_{m=1}^{\infty} b_m$$

We multiply with 691 and get

$$(65520 + 1008 \cdot 691)\tau(m) - 65520\sigma_{11}(m) = -691b_m$$

and finally

$$65520(\tau(m) - \sigma_{11}(m)) = 691 \underbrace{(-b_m - 1008)}_{\in \mathbb{Z}}$$

But 691 is a prime and  $65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$  is not divisible by 691. Therefore  $\tau(m) - \sigma_{11}(m)$  is divisible by 691.

Exercise 5

Remark: I will use  $\widehat{f}(y) = \int f(x)e^{-2\pi ixy} dx$ , as in Koblitz. It is for this reason why the factor  $\chi(-1)$  in a.) does not appear.

a.) The Idea is to use the Poisson Summation formula and the periodicity of  $\chi$ . We define therefore  $f_n(x) := f(Nx + n)$ ,  $0 \leq n \leq N - 1$ . It follows

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \chi(n)f(n) &= \sum_{m=0}^{N-1} \sum_{n=-\infty}^{\infty} \chi(nN + m)f(nN + m) = \sum_{m=0}^{N-1} \left( \chi(m) \sum_{n=-\infty}^{\infty} \right) f_m(n) \\ &= \sum_{m=0}^{N-1} \left( \chi(m) \sum_{n=-\infty}^{\infty} \widehat{f}_m(n) \right) \end{aligned}$$

Notice that the first sum is absolutely convergent since  $x^2 f(x) \leq 1$  for  $x$  big enough and so we are allowed to change the order of summation. Now we use  $\widehat{f}_m(n) = \widehat{f}(m+nN) = \frac{1}{N} e^{2\pi i nm/N} \widehat{f}\left(\frac{n}{N}\right)$ . This can be seen for example by direct calculation. We get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \chi(n)f(n) &= \sum_{m=0}^{N-1} \left( \chi(m) \sum_{n=-\infty}^{\infty} \frac{1}{N} e^{2\pi i nm/N} \widehat{f}\left(\frac{n}{N}\right) \right) = \frac{1}{N} \sum_{n=-\infty}^{\infty} \left( \widehat{f}\left(\frac{n}{N}\right) \sum_{m=0}^{N-1} \chi(m) e^{2\pi i nm/N} \right) \\ &\stackrel{(i)}{=} \frac{1}{N} \sum_{n=-\infty}^{\infty} \widehat{f}\left(\frac{n}{N}\right) \overline{\chi(n)} \tau(\chi) = \frac{\tau(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \widehat{f}\left(\frac{n}{N}\right) \end{aligned}$$

b.) We set  $f(x) := e^{-\pi tx^2}$ . Then  $\widehat{f}(y) = \sqrt{\frac{1}{t}} e^{-\pi y^2/t}$ . If we apply this to a.) we get

$$2\Theta_\chi(t) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi tn^2} = \frac{\tau(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \sqrt{\frac{1}{t}} e^{-\pi(\frac{n}{N})^2/t} = 2 \frac{\tau(\chi)}{N\sqrt{t}} \Theta_{\overline{\chi}}\left(\frac{1}{tN^2}\right)$$

c.) Notice  $\chi(-n) e^{-\pi(-n)^2 t} = \chi(-1) \chi(n) e^{-\pi n^2 t}$ . Therefore we have to use an other  $\Theta$  for  $\chi(-1) = -1$

We will use again a.), but with  $f(x) = x e^{-\pi tx^2}$ . Therefore we need to calculate the fourier transformation of  $f$ . Let  $T > 0$  be arbitrary. Then

$$\int_T^T \underbrace{se^{-\pi ts^2}}_{u'} \underbrace{e^{-2\pi iys}}_v ds = \left[ \frac{-1}{2\pi t} e^{-\pi ts^2} e^{-2\pi iys} \right]_{s=-T}^T - \int_T^T \frac{-1}{2\pi t} e^{-\pi ts^2} (-2\pi iy) e^{-2\pi iys}$$

We get

$$\widehat{f}(y) = \lim_{T \rightarrow \infty} \int_{-T}^T se^{-\pi ts^2} e^{-2\pi iys} ds = \frac{-iy}{t} \int_{-\infty}^{\infty} e^{-\pi ts^2} e^{-2\pi iys} ds = \frac{-iy}{t} \sqrt{\frac{1}{t}} e^{-\pi y^2/t} = \frac{-iy}{t^{3/2}} e^{-\pi y^2/t}$$

As in b.)

$$\begin{aligned} 2\Theta_\chi(t) &= \sum_{n=-\infty}^{\infty} n \chi(n) e^{-\pi tn^2} = \frac{\tau(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \frac{-in}{Nt^{3/2}} e^{-\pi(\frac{n}{N})^2/t} = \frac{-i\tau(\chi)}{N^2 t^{3/2}} \sum_{n=-\infty}^{\infty} n \overline{\chi(n)} e^{-\pi(\frac{n}{N})^2/t} \\ &= 2 \frac{-i\tau(\chi)}{N^2 t^{3/2}} \Theta_{\overline{\chi}}(t) \end{aligned}$$

d.) As usual we write  $s = \sigma + it$ . For shortness we write  $\Theta = \Theta_\chi$  and  $\Theta_c = \Theta_{\overline{\chi}}$

Claim  $f(s) = \int_0^\infty \Theta(x) x^{s-1} dx$  is an entire function

Proof Let  $K \subset \mathbb{C}$  be compact. We have

$$\int_0^\infty |\Theta(x) x^{s-1}| dx = \underbrace{\int_0^1 \Theta(x) x^{\sigma-1} dx}_{(1)} + \underbrace{\int_1^\infty \Theta(x) x^{\sigma-1} dx}_{(2)}$$

Existence of (2): We set  $T := \max\{2, \sup_{s \in K} \sigma\}$

Notice  $\chi(0) = 0$  and  $|\chi(n)| = 1$

$$\begin{aligned} \int_1^\infty \Theta(x) x^{\sigma-1} dx &\leq \int_1^\infty \Theta(x) x^{T-1} dx \leq \int_1^\infty \sum_{n=1}^\infty n e^{-\pi n^2 x} x^{T-1} dx \\ &\leq \sum_{n=1}^\infty \int_{1/\pi n^2}^\infty n e^{-y} \left(\frac{y}{\pi n^2}\right)^{T-1} \frac{dy}{\pi n^2} \leq \pi^{-T} \sum_{n=1}^\infty \frac{1}{n^{2T-1}} \int_0^\infty e^{-\pi y} y^{T-1} dy \\ &= \pi^{-T} \Gamma(T) \zeta(2T-1) \end{aligned}$$

Existence of (1): We set  $y = \frac{1}{N^2x}$  and get

$$\int_0^1 \Theta(x)x^{\sigma-1}dx = \int_{1/N^2}^{\infty} \Theta\left(\frac{1}{N^2y}\right) \frac{1}{y^{\sigma+1}} \frac{1}{N^{\sigma}} dy = C(\epsilon) \int_{1/N^2}^{\infty} \Theta_c(y)y^{1/2+\epsilon} \frac{1}{y^{\sigma+1}} \frac{1}{N^{\sigma}} dy$$

Now we can proceed as for (2) □

Claim  $\Lambda(2s - \epsilon, \chi) = \int_0^{\infty} \Theta(x)x^{s-1}dx$  for  $\sigma$  big enough.

Proof Case 1:  $\chi(-1) = 1$ . We get

$$\begin{aligned} \int_0^{\infty} \Theta(x)x^{s-1}dx &= \int_0^{\infty} \sum_{n=1}^{\infty} \chi(n)e^{-\pi n^2x}x^{s-1}dx = \sum_{n=1}^{\infty} \chi(n) \int_0^{\infty} e^{-\pi n^2x}x^{s-1}dx \\ &= \sum_{n=1}^{\infty} \chi(n) \int_0^{\infty} e^{-y} \left(\frac{y}{\pi n^2}\right)^{s-1} \frac{dy}{\pi n^2} = \pi^{-s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{2s}} \Gamma(s) \\ &= \pi^{-s} \Gamma(s) L(2s, \chi) = \Lambda(2s, \chi) \end{aligned}$$

Case 2:  $\chi(-1) = -1$ . We use  $-n\chi(-n) = n\chi(n)$  and get

$$\begin{aligned} \int_0^{\infty} \Theta(x)x^{s-1}dx &= \int_0^{\infty} \sum_{n=1}^{\infty} n\chi(n)e^{-\pi n^2x}x^{s-1}dx = \sum_{n=1}^{\infty} n\chi(n) \int_0^{\infty} e^{-\pi n^2x}x^{s-1}dx \\ &= \sum_{n=1}^{\infty} n\chi(n) \int_0^{\infty} e^{-y} \left(\frac{y}{\pi n^2}\right)^{s-1} \frac{dy}{\pi n^2} = \pi^{-s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{2s-1}} \Gamma(s) \\ &= \pi^{-s} \Gamma(s) L(2s - 1, \chi) = \Lambda(2s - 1, \chi) \end{aligned}$$

□

### Functional Equation

$$\begin{aligned} \frac{(-i)^{\epsilon} \tau(\chi)}{N^{1+\epsilon}} \Lambda(2s - \epsilon, \bar{\chi}) &= \frac{(-i)^{\epsilon} \tau(\chi)}{N^{1+\epsilon}} \int_0^{\infty} \Theta_c(x)x^{s-1}dx = \frac{(-i)^{\epsilon} \tau(\bar{\chi})}{N^{1+\epsilon}} \int_0^{\infty} \Theta_c\left(\frac{1}{N^2y}\right) \frac{1}{N^{2s}} \frac{dy}{y^{s+1}} \\ &= \frac{1}{N^{2s}} \int_0^{\infty} \Theta(y)y^{(1/2+\epsilon-s)-1} ds = \frac{1}{N^{2s}} \Lambda(1 + \epsilon - 2s, \chi) \end{aligned}$$

Now we set  $s = \frac{-s+1+\epsilon}{2}$  and get

$$\Lambda(s, \chi) = (-i)^{\epsilon} \tau(\chi) \frac{N^{-s+1+\epsilon}}{N^{1+\epsilon}} \Lambda(1 - s, \bar{\chi}) = (-i)^{\epsilon} \tau(\chi) N^{-s} \Lambda(1 - s, \bar{\chi})$$