RIEMANN'S FIRST PROOF OF THE ANALYTIC CONTINUATION OF $\zeta(s)$ AND $L(s,\chi)$

FELIX RUBIN
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Abstract. In this chapter, we will see a proof of the analytic continuation of the Riemann zeta function $\zeta(s)$ and the Dirichlet L function $L(s,\chi)$ via the Hurwitz zeta function. This then gives rise to a functional equation for $\zeta(s)$ and a direct computation for the value of this function at negative integer points.

1. The Hurwitz zeta function

We have already seen the definition of the **Riemann zeta function** $\zeta(s)$ and the **Dirichlet L function** $L(s,\chi)$ as series:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \qquad \text{and} \qquad L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

where $s=\sigma+it\in\mathbb{C}$, $\sigma>1$, $t\in\mathbb{R}$ and χ is a Dirichlet character modulo k, $k\in\mathbb{N}$. These notations will be used throughout this chapter.

We can unify the treatment of the above two functions by introducing the Hurwitz zeta function:

Definition 1.1. For $\sigma>1$, we define $\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, 0<a\leq 1$, the **Hurwitz zeta function**.

Remark 1.2.
(1) For $a=1$, $\zeta(s,1)=\zeta(s)$.
(2) $L(s,\chi)=k^{-s}\sum_{r=1}^{k}\chi(r)\zeta(s,\frac{r}{k})$, if χ is a Dirichlet character modulo k.

The second statement in the remark follows from the following calculation: Write $n=qk+r$, $1\leq r\leq k$, $q=0,1,2,....$ We then have:

$$\begin{aligned}L(s,\chi) &= \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\sum_{r=1}^{k}\sum_{q=0}^{\infty}\frac{\chi(qk+r)}{(qk+r)^s}=\frac{1}{k^s}\sum_{r=1}^{k}\chi(r)\sum_{q=0}^{\infty}\frac{1}{(q+\frac{r}{k})^s}\\ &= k^{-s}\sum_{r=1}^{k}\chi(r)\zeta(s,\frac{r}{k}).\end{aligned}$$

Theorem 1.3. *The series $\zeta(s,a)$ converges absolutely for $\sigma>1$ with uniform convergence in every half-plane $\sigma>1+\delta$, $\delta>0$. Hence, $\zeta(s,a)$ is an analytic function of s in the half-plane $\sigma>1$.*

Proof. We have

$$\sum_{n=0}^{\infty}|(n+a)^{-s}|=\sum_{n=0}^{\infty}(n+a)^{-\sigma}\leq\sum_{n=0}^{\infty}(n+a)^{-(1+\delta)}$$

where the first equality follows from $|x^s|=|e^{s\ln x}|=|e^{\sigma\ln x}e^{it\ln x}|=e^{\sigma\ln x}=x^{\sigma}$, for $x\in\mathbb{C}$, and the theorem follows. □

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We will be able to analytically continue the Hurwitz zeta function beyond the line $\sigma = 1$ by an integral representation of $\zeta(s, a)$ obtained from the integral formula for the Gamma function.

We need the following properties of the Gamma function:

For $\sigma > 0$, we can define the **Gamma function** by $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$. This function can be analytically continued to the whole complex plane with simple poles at $0, -1, -2, -3, ...$ with residues $\frac{(-1)^n}{n!}$ at $s = -n, n \in \mathbb{N}$. Moreover, the gamma function satisfies the functional equations $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$. Finally, we note that $\Gamma(n+1) = n!$, if $n \in \mathbb{N}$. For more detail on the Gamma function, see for example [2].

The next theorem gives an integral representation for the Hurwitz zeta function.

Theorem 1.4. *For $\sigma > 1$ and $0 < a \leq 1$, we have*

$$\Gamma(s)\zeta(s, a) = \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx \tag{1.1}$$

Proof. First, keep s real, $s > 1$ and later extend it to complex s by analytic continuation.

By a change of variable $x = (n+a)t$, $n \geq 0$, we have for the Gamma function:

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx = (n+a)^s \int_0^\infty e^{-(n+a)t}t^{s-1}dt$$
$$\Rightarrow (n+a)^{-s}\Gamma(s) = \int_0^\infty e^{-nt}e^{-at}t^{s-1}dt$$

Summing over all n, this gives: $\zeta(s, a)\Gamma(s) = \sum_{n=0}^\infty \int_0^\infty e^{-nt}e^{-at}t^{s-1}dt$, where the left hand side converges for $\sigma > 1$. By applying Beppo-Levi's theorem (1.5 below), we can interchange the sum and the integral to obtain: $\zeta(s, a)\Gamma(s) = \int_0^\infty \sum_{n=0}^\infty e^{-nt}e^{-at}t^{s-1}dt$.

Since $t > 0$, $0 < e^{-t} < 1$ and thus $\sum_{n=0}^\infty e^{-nt} = \frac{1}{1-e^{-t}}$. This gives: $\sum_{n=0}^\infty e^{-nt}e^{-at}t^{s-1} = \frac{e^{-at}t^{s-1}}{1-e^{-t}}$, for almost all $t \in [0, \infty)$. Hence:

$$\zeta(s, a)\Gamma(s) = \int_0^\infty \frac{e^{-at}t^{s-1}}{1-e^{-t}}dt, \quad \textit{where} \quad s > 1, \quad s \in \mathbb{R}.$$

In order to extend this formula to the half plane $\sigma > 1$, note that the left hand side of the equation is analytic for $\sigma > 1$. To show that the right hand side is analytic, we assume that for some $\delta > 0$, $c > 1$, we have $1 + \delta \leq \sigma \leq c$. Because $t^{\sigma-1} \leq t^\delta$ for $t \in [0, 1]$, $t^{\sigma-1} \leq t^{c-1}$, for $t \geq 1$ and $e^t - 1 \geq t$, for $t \geq 0$, this gives:

$$\begin{aligned}
\int_0^\infty \left|\frac{e^{-at}t^{s-1}}{1-e^{-t}}\right|dt &\leq \int_0^\infty \frac{e^{-at}t^{\sigma-1}}{1-e^{-t}}dt = \left(\int_0^1 + \int_1^\infty\right)\frac{e^{-at}t^{\sigma-1}}{1-e^{-t}}dt \\
\int_0^1 \frac{e^{-at}t^{\sigma-1}}{1-e^{-t}}dt &\leq \int_0^1 \frac{e^{(1-a)t}t^\delta}{e^t-1}dt \leq e^{1-a}\int_0^1 t^{\delta-1}dt = \frac{e^{1-a}}{\delta}, \quad \text{and} \\
\int_1^\infty \frac{e^{-at}t^{\sigma-1}}{1-e^{-t}}dt &\leq \int_1^\infty \frac{e^{-at}t^{c-1}}{1-e^{-t}}dt \leq \int_0^\infty \frac{e^{-at}t^{c-1}}{1-e^{-t}}dt = \Gamma(c)\zeta(c, a).
\end{aligned}$$

Hence, the integral in (1.1) converges uniformly in every strip $1 + \delta \leq \sigma \leq c$ and therefore represents an analytic function in the half plane $\sigma > 1$. From this, the theorem follows. □

Beppo-Levi's theorem can be found in any course on measure and integration theory or probability theory and says the following:

Theorem 1.5. *(**Beppo-Levi**) Let $\{f_n\}_{n\in\mathbb{N}}$ be an increasing sequence of non-negative, measurable functions from any space Ω into $\mathbb{R}$ (i.e. $0 \le f_1 \le f_2 \le f_3 \le ...$, almost everywhere). Then,*

$$\lim_{n\to\infty}\int_\Omega f_n(\omega)d\mu(\omega) = \int_\Omega \lim_{n\to\infty} f_n(\omega)d\mu(\omega),$$

where μ denotes the measure on the space Ω.

We need another representation of $\zeta(s,a)$, in order to give its analytic continuation beyond $\sigma = 1$. Consider the contour C, which is a loop around the negative real axis as shown in figure 1. We decompose C into the parts C_1, C_2 and C_3. C_2 is a circle around 0 with radius $c < 2\pi$ (pos. oriented). C_1 and C_3 are the lower and upper edges of a "cut" in the complex plane along the negative real axis. We use the parameterizations $z = re^{-\pi i}$ on C_1, $z = re^{\pi i}$ on C_3 with $r \in [c,\infty)$ and $z = ce^{i\theta}$ with $\theta \in [-\pi,\pi]$ on C_2.

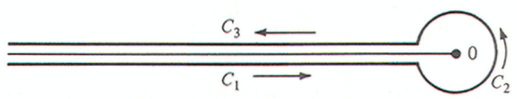

FIGURE 1. The contour C around the negative real axis

The next result allows us to represent the Hurwitz zeta function as an integral along C combined with the Gamma function, which will lead us to the analytic continuation in the next section.

Theorem 1.6. *If $0 < a \le 1$, $\sigma > 1$, the following holds*

$$\zeta(s,a) = \Gamma(1-s)I(s,a), \tag{1.2}$$

where $I(s,a) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}e^{az}}{1-e^z}dz$ is an entire function of s.

Note here, that for the definition of z^{s-1} in the integrand, we write $e^{(s-1)\ln(z)}$ and use the principal branch of the logarithm which is defined on $\mathbb{C}\backslash\{x\in\mathbb{R}; x\le 0\}$. This branch does make a jump of $2\pi i$ when crossing the negative real axis. Therefore if we continue the logarithm analytically from above the negative real axis and from below the negative real axis to this axis and take the integral along the contour C, this integral will not cancel out in general and moreover, the integral does not depend on the radius c, as long as $c \le 2\pi$.

Proof. The integral along C_2 is an entire function in s since the integrand is an entire function ($1 - e^z \ne 0$ on C_2).We thus only need to prove that the integrals along C_1 and C_3 converge uniformly on any disk $|s| \le M$, $M > 0$.

Along C_1, we have for $r \ge 1$:

$$|z^{s-1}| = r^{\sigma-1}|e^{-\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{\pi t} \le r^{M-1}e^{\pi M} \quad \text{since} \quad |s| \le M.$$

Similarly on C_3:

$$|z^{s-1}| = r^{\sigma-1}|e^{\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{-\pi t} \le r^{M-1}e^{\pi M} \quad \text{since} \quad |s| \le M.$$

We thus have for $r \ge 1$ on C_1 and C_3:
$|\frac{z^{s-1}e^{az}}{1-e^z}| \le \frac{r^{M-1}e^{\pi M}e^{-ar}}{1-e^{-r}} = \frac{r^{M-1}e^{\pi M}e^{(1-a)r}}{e^r-1}$. But $e^r - 1 > \frac{e^r}{2}$, when $r > \ln 2$, hence the integrand is bounded by $Ar^{M-1}e^{-ar}$, where the constant A only depends on M. But $\int_c^\infty r^{M-1}e^{-ar}dr$ converges for $c > 0$, and hence the integral does converge uniformly on every disk $|s| \le M$. Therefore, $I(s,a)$ is an entire function of s.

To prove (1.2), we write $2\pi i I(s,a) = (\int_{C_1} + \int_{C_2} + \int_{C_3}) z^{s-1} g(z) dz$, where $g(z) = \frac{e^{az}}{1-e^z}$. On C_1 and C_3, $g(z) = g(-r)$ and on C_2, we write $z = ce^{i\theta}$, with $-\pi \leq \theta \leq \pi$.

$$\begin{aligned} \Rightarrow 2\pi i I(s,a) &= \int_{\infty}^{c} r^{s-1} e^{-\pi i s} g(-r) dr + i \int_{-\pi}^{\pi} c^{s-1} e^{(s-1)i\theta} c e^{i\theta} g(ce^{i\theta}) d\theta \\ &+ \int_{c}^{\infty} r^{s-1} e^{\pi i s} g(-r) dr \\ &= 2i \sin(\pi s) \int_{c}^{\infty} r^{s-1} g(-r) dr + i c^s \int_{-\pi}^{\pi} e^{is\theta} g(ce^{i\theta}) d\theta. \end{aligned}$$

Now, let $c \to 0$: We have $\lim_{c \to 0} \int_c^{\infty} r^{s-1} g(-r) dr = \Gamma(s)\zeta(s,a)$ for $\sigma > 1$, according to equation (1.1). It remains to show that the second integral from $-\pi$ to π goes to zero when $c \to 0$ in order to prove (1.2).

Note that $g(z)$ is analytic in $|z| < 2\pi$ except for a simple pole at $z = 0$. Hence, $zg(z)$ is analytic everywhere in $|z| < 2\pi$ and therefore bounded by some constant A. This gives for $\sigma > 1$:

$$|c^s \int_{-\pi}^{\pi} e^{is\theta} g(ce^{i\theta}) d\theta| \leq c^{\sigma} \int_{-\pi}^{\pi} e^{-t\theta} \frac{A}{c} d\theta \leq 2\pi A e^{\pi|t|} c^{\sigma-1} \to 0 \quad when \quad c \to 0$$

$$\Rightarrow \pi I(s,a) = \sin(\pi s)\Gamma(s)\zeta(s,a)$$

and the result follows from $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$. □

2. The analytic continuation of the Hurwitz zeta function

We can use equality (1.2) to extend the definition of the Hurwitz zeta function beyond the line $\sigma = 1$, since the right hand side of this equation is meaningful for any $s \in \mathbb{C}$. We therefore define

$$\zeta(s,a) := \Gamma(1-s) I(s,a), \quad \forall s \in \mathbb{C}, \quad 0 < a \leq 1.$$

Theorem 2.1. *The function $\zeta(s,a)$ so defined is analytic on $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$ with residue 1.*

Proof. We know that $I(s,a)$ is entire, hence any possible singularities come from $\Gamma(1-s)$. $\Gamma(1-s)$ has poles at $s = 1, 2, 3, \ldots$. However, we have seen that $\zeta(s,a)$ is analytic at $s = 2, 3, \ldots$, so $s = 1$ is the only possible pole of $\zeta(s,a)$. For $s = n \in \mathbb{N}$,

$$I(n,a) = \frac{1}{2\pi i} \int_C \frac{z^{n-1} e^{az}}{1-e^z} dz = \frac{1}{2\pi i} \int_{C_2} \frac{z^{n-1} e^{az}}{1-e^z} dz = Res_{z=0} \left(\frac{z^{n-1} e^{az}}{1-e^z} \right)$$

since the integrals along C_1 and C_3 cancel out. Hence, for $n = 1$,

$$I(1,a) = Res_{z=0} \left(\frac{e^{az}}{1-e^z} \right) = \lim_{z \to 0} \frac{z e^{az}}{1-e^z} = \lim_{z \to 0} \frac{z}{1-e^z} = -1$$

Now, we compute

$$\begin{aligned} \lim_{s \to 1} (s-1)\zeta(s,a) &= -\lim_{s \to 1} (1-s)\Gamma(1-s) I(s,a) \\ &= -I(1,a) \lim_{s \to 1} \Gamma(2-s) = \Gamma(1) = 1. \end{aligned}$$

Therefore, $\zeta(s,a)$ has a simple pole at $s = 1$ with residue 1. □

By the equations given in remark 1.2, we can now give analytic continuations of the Riemann zeta and the Dirichlet L functions by using these equations as definitions of the respective functions.

Theorem 2.2. (1) *$\zeta(s)$ is analytic on $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1.*

(2) *For the principal character χ_1 modulo k, $L(s,\chi_1)$ is analytic on $\mathbb{C}$ with a simple pole at $s = 1$ with residue $\frac{\phi(k)}{k}$.*

(3) *If $\chi \neq \chi_1$, $L(s,\chi)$ is an entire function of s.*

$\phi(k)$ denotes **Euler's totient** which gives the number of elements $n \leq k$, $n \in \mathbb{N}$, such that $(k,n) = 1$.

Proof. The first statement is trivial. To prove the other two, we note that

$$\sum_{r=1}^{k} \chi(r) = \begin{cases} 0 & \text{if} \quad \chi \neq \chi_1 \\ \phi(k) & \text{if} \quad \chi = \chi_1 \end{cases}$$

This formula is a consequence of the definition of Dirichlet characters of reduced residue classes (see for example [3]). Now, $\zeta(s, \frac{r}{k})$ has a simple pole at $s = 1$ with residue 1. Hence $\chi(r)\zeta(s, \frac{r}{k})$ has a simple pole at $s = 1$ with residue $\chi(r)$.

$$\begin{aligned} \Rightarrow Res_{s=1}(L(s,\chi)) &= \lim_{s\to 1}(s-1)L(s,\chi) = \lim_{s\to 1}(s-1)k^{-s}\sum_{r=1}^{k}\chi(r)\zeta(s,\frac{r}{k}) \\ &= \frac{1}{k}\sum_{r=1}^{k}\chi(r) = \begin{cases} 0 & \text{if} \quad \chi \neq \chi_1 \\ \frac{\phi(k)}{k} & \text{if} \quad \chi = \chi_1 \end{cases} \end{aligned}$$

□

3. Evaluation of $\zeta(-n,a)$ for $n \in \mathbb{N}$

We can now explicitly calculate $\zeta(-n,a)$ for $n \in \mathbb{N}$: $\zeta(-n,a) = \Gamma(1+n)I(-n,a) = n!I(-n,a)$. In the proof of theorem 2.1 we have seen that for $n \in \mathbb{N}$, $I(-n,a) = Res_{z=0}(\frac{z^{-n-1}e^{az}}{1-e^z})$. The calculation of this residue leads to the Beroulli polynomials.

Definition 3.1. $\forall x \in \mathbb{C}$, we define the **Bernoulli polynomials** $B_n(x)$ by the equation $\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n$, where $|z| < 2\pi$.

This leads to the explicit result

Theorem 3.2. *$\forall n \in \mathbb{N}$ and $0 < a \leq 1$, we have $\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}$.*

Proof. We already know that $\zeta(-n,a) = n!I(-n,a)$. But

$$\begin{aligned} I(-n,a) &= Res_{z=0}\left(\frac{z^{-n-1}e^{az}}{1-e^z}\right) = -Res_{z=0}\left(z^{-n-2}\frac{ze^{az}}{e^z-1}\right) \\ &= -Res_{z=0}\left(z^{-n-2}\sum_{m=0}^{\infty}\frac{B_m(a)}{m!}z^m\right) = -\frac{B_{n+1}(a)}{(n+1)!} \end{aligned}$$

Hence, the theorem follows. □

4. A functional equation for the Riemann zeta function

Here, we give e functional equation for $\zeta(s)$, which leads us to the so called trivial zeros of the Riemann zeta function.

Definition 4.1. For $x \in \mathbb{R}$ and $\sigma > 1$, define the Dirichlet series

$$F(x,s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}$$

$F(x,s)$ is called the **periodic zeta function**.

Note that $F(x,s)$ is periodic in x with period 1 and $F(1,s) = \zeta(s)$. Moreover, the series converges absolutely for $\sigma > 1$.

Proposition 4.2. If $0 < a \leq 1$ and $\sigma > 1$, the **Hurwitz formula** holds:

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\frac{\pi i s}{2}} F(a,s) + e^{\frac{\pi i s}{2}} F(-a,s)\right)$$

In the proof of this proposition, we need the following lemma:

Lemma 4.3. For $0 < r < \pi$, denote with $S(r)$ the region

$$S(r) = \mathbb{C} \setminus \left(\bigcup_{n \in \mathbb{Z}} B(2n\pi i, r) \right),$$

where $B(c,r)$ denotes the open disk with center c and radius r. Then, if $0 < a \leq 1$, the function $g(z) = \frac{e^{az}}{1-e^z}$ is bounded in $S(r)$. (The bound depends on r.)

For a proof of this lemma see [1]. We now turn to the proof of Hurwitz's formula.

Proof. Consider the function $I_N(s,a) = \frac{1}{2\pi i}\int_{C(N)} \frac{z^{s-1}e^{az}}{1-e^z}dz$, where $C(N)$ is the contour shown in figure 2, and $N \in \mathbb{N}$. Note that we use again the principal branch of the logarithm to define z^{s-1} in the integrand, as we did in the proof of formula (1.2).

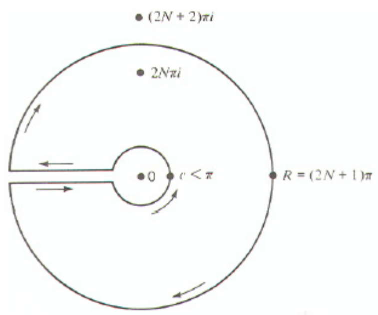

FIGURE 2

We now show that if $\sigma < 0$, the integral along the outer circle tends to 0 as $N \to \infty$. Hence, $I_N(s,a) \to I(s,a)$, for $N \to \infty$ and $\sigma < 0$. Parameterizing $z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, we have

$$|z^{s-1}| = |R^{s-1}e^{i\theta(s-1)}| = R^{\sigma-1}e^{-t\theta} \leq R^{\sigma-1}e^{\pi|t|}.$$

Since the outer circle lies in the set $S(r)$ of the preceding lemma, the absolute value of the integrand is bounded by $Ae^{\pi|t|}R^{\sigma-1}$, A being the bound for $|g(z)|$ implied by the lemma. Hence, the absolute value of the integral along the outer circle is bounded by $2\pi Ae^{\pi|t|}R^{\sigma}$, which tends to 0 as $R \to \infty$, for $\sigma < 0$. We thus have:

$$\lim_{N\to\infty} I_N(1-s,a) = I(1-s,a) \quad \text{if} \quad \sigma > 1.$$

On the other hand, we have by Cauchy's residue theorem:

$$\begin{aligned} I_N(1-s,a) &= -\sum_{n=-N,n\neq 0}^{N} Res_{z=2n\pi i}\left(\frac{z^{-s}e^{az}}{1-e^z}\right) \\ &= -\sum_{n=1}^{N}\left(Res_{z=2n\pi i}\left(\frac{z^{-s}e^{az}}{1-e^z}\right)+Res_{z=-2n\pi i}\left(\frac{z^{-s}e^{az}}{1-e^z}\right)\right) \end{aligned}$$

Now

$$Res_{z=2n\pi i}\left(\frac{z^{-s}e^{az}}{1-e^z}\right)=\lim_{z\to 2n\pi i}(z-2n\pi i)\frac{z^{-s}e^{az}}{1-e^z}=-\frac{e^{2n\pi i a}}{(2n\pi i)^s},$$

hence $I_N(1-s,a)=\sum_{n=1}^{N}\frac{e^{2n\pi i a}}{(2n\pi i)^s}+\sum_{n=1}^{N}\frac{e^{-2n\pi i a}}{(-2n\pi i)^s}$. But $i^{-s}=e^{\frac{-\pi i s}{2}}$ and $(-i)^{-s}=e^{\frac{\pi i s}{2}}$ so

$$I_N(1-s,a)=\frac{e^{\frac{-\pi i s}{2}}}{(2\pi)^s}\sum_{n=1}^{N}\frac{e^{2n\pi i a}}{n^s}+\frac{e^{\frac{\pi i s}{2}}}{(2\pi)^s}\sum_{n=1}^{N}\frac{e^{-2n\pi i a}}{n^s}.$$

Letting $N\to\infty$ and multiplying both sides with $\Gamma(s)$, we obtain the desired result. □

We can now state the functional equation for the Riemann zeta function.

Theorem 4.4. *For all $s\in\mathbb{C}$, we have*

$$\begin{aligned} \zeta(1-s) &= 2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s) \\ \textit{or equivalently}\quad \zeta(s) &= 2(2\pi)^{s-1}\Gamma(1-s)\sin(\frac{\pi s}{2})\zeta(1-s). \end{aligned}$$

Proof. Take $a=1$ in Hurwitz's formula to obtain for $\sigma>1$:

$$\begin{aligned} \zeta(1-s) &= \zeta(1-s,1)=\frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\frac{\pi i s}{2}}F(1,s)+e^{\frac{\pi i s}{2}}F(1,s)\right) \\ &= \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\frac{\pi i s}{2}}\zeta(s)+e^{\frac{\pi i s}{2}}\zeta(s)\right)=\frac{\Gamma(s)}{(2\pi)^s}2\cos(\frac{\pi s}{2})\zeta(s) \end{aligned}$$

Hence, the theorem is proved for $\sigma>1$. The result follows by analytic continuation. □

Taking $s=2n+1$, $n\in\mathbb{N}$, we find that the factor $\cos(\frac{\pi s}{2})$ vanishes and we get the **trivial zeros** of $\zeta(s)$: $\zeta(-2n)=0$, $\forall n\in\mathbb{N}$.

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