Introduction

This talk contains three parts

Part 1: The main goal of this part is to proof $(2\pi)^{-12}\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi iz}$. To do this, we start we the non absolutely convergent Eisenstein Series $E_2(z)$ and their behavior under Γ. As next we define the Dedekind *eta* function. With the help of $E_2(z)$ we can show $\eta(\frac{-1}{z}) = \sqrt{(z/i)}\eta(z)$. The theorem then directly follows from this.

Part 2: The target of this part is the q-Expansion of η, i.e. $\eta(z) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}$. This directly follows from the Euler pentagonal number theorem. We will gave for this the idea of three different proofs: With modular forms, with elliptic functions and a combinatorial one.

Part 3: We will show, that the coefficients of the j-function are integral and gave an idea how one can show

- $j(2n) \equiv 0 \ (mod\ 2^{11})$
- $j(2n) \equiv 0 \ (mod\ 3^5)$
- $j(2n) \equiv 0 \ (mod\ 5^2)$
- $j(2n) \equiv 0 \ (mod\ 7)$

1 Infinite product of $\Delta(z)$

Satz 1.1. $(2\pi)^{-12}\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi i z}$

pro memoria:

$$\Delta = \frac{(2\pi)^{12}}{1728}(E_4^3 - E_6^2), E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, E_6 = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n$$

$$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z} \quad \texttt{the Riemannian } \zeta\texttt{-fkt, with } \zeta(2) = \frac{\pi^2}{6}$$

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right), \quad \frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2}$$

To proof Satz 1.1., we will first introduce $E_2(z)$. Notice that E_2 is not absolutely convergent.

Definition 1.2.

$$E_2(z) = \frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^2} \tag{1}$$

$$= 1 + \frac{3}{\pi^2}\sum_{m\neq 0}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^2} \tag{2}$$

$$= 1 + \frac{6}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^2} \tag{3}$$

Lemma 1.3. *$E_2(z)$ is holomorphic at $\mathbb{H}\cup\{\infty\}$ and*

$$E_2(z+1) = E_2(z) \tag{4}$$

$$z^{-2}E_2\left(\frac{-1}{z}\right) = E_2(z) + \frac{12}{2\pi i z} \tag{5}$$

Proof

The Equation (4) follows from (2) or (3).

to show that $E_2(z)$ is holomorphic, we will write the inner sum of (2) in a different way. We will use $|q| < 1, \forall z \in \mathbb{H}$

$$\pi\cot(\pi z) = \pi i\frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i\sum_{n=0}^{\infty}q^n$$

Now we differentiate, multiply by -1 and replace z with mz ($m > 0$)

$$(2\pi i)^2 \sum_{n=1}^{\infty} nq^{nm} = \frac{\pi^2}{sin^2(\pi mz)} = \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2}$$

We replace the inner sum of (3) we get a new form of $E_2(z)$.

$$E_2(z) = 1 - 24 \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} dq^{dm} \tag{6}$$

With the follow calculation we get that (6) is absolutely convergent.

$$\left| \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} d(q^d)^m \right| \leq \sum_{d=1}^{\infty} \left| d \frac{q^d}{1-q^d} \right| \leq^{(a)} C + 2 \sum_{d=1}^{\infty} \left| dq^d \right| \leq^{(b)} C + 2 \sum_{d=1}^{\infty} |q|^{\frac{d}{2}} < \infty$$

We have used
(a) $|1 - q^d| \geq \frac{1}{2}$ except for finitely many d.
(b) $d|q|^{d/2} \leq 1$ except for finitely many d, since $|q|^{d/2}$ goes exponentially fast to 0.
So $E_2(z)$ ist holomorphic at $\mathbb{H}$. Equation (6) also shows, that $E_2(z)$ is holomorphic at ∞.
It rests to show (5). We want to exchange the two sums of (5) and to do that we 'correct' the double sum with terms to make it absolutely convergent.

$$a_{m,n} = a_{n,m} = \frac{1}{(mz+n-1)(mz+n)} = \frac{1}{mz+n-1} - \frac{1}{mz+n} \tag{7}$$

Because $\frac{1}{(mz+n)^2} - a_{m,n} \approx \frac{1}{(mz+n)^3}$ we get that the following 'new' E_2 is absolutely convergent.

$$\begin{aligned} \widetilde{E_2}(z) &= 1 + \frac{3}{\pi^2} \sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \left(\frac{1}{(mz+n)^2} - a_{m,n}(z) \right) \\ &= 1 + \frac{3}{\pi^2} \sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2} + \frac{3}{\pi^2} \sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \underbrace{\left(\frac{1}{mz+n} - \frac{1}{mz+n-1} \right)}_{Telescopsum} = E_2(z) \end{aligned}$$

But now we can exchange the two sums in $\widetilde{E_2}(z)$:

$$\begin{aligned}
\widetilde{E_2}(z) &= 1+\frac{3}{\pi^2}\sum_{n=-\infty}^{\infty}\sum_{m\neq 0}\left(\frac{1}{(mz+n)^2}-a_{m,n}(z)\right)\\
&= 1+\frac{3}{\pi^2 z^2}\sum_{m=-\infty}^{\infty}\sum_{n\neq 0}\left(\frac{1}{(m\left(\frac{-1}{z}\right)+n)^2}\right)-\overbrace{\frac{3}{\pi^2}\sum_{n=-\infty}^{\infty}\sum_{m\neq 0}a_{m,n}(z)}^{S:=}\\
&= 1+\frac{3}{\pi^2 z^2}\sum_{m\neq 0}\sum_{n\neq 0}(...)+\frac{3}{\pi^2 z^2}\underbrace{\sum_{m=0}\sum_{n\neq 0}\frac{1}{n^2}}_{=\frac{\pi^2}{3}}-S\\
&= \frac{1}{z^2}+\frac{3}{\pi^2 z^2}\sum_{m\neq 0}\left(\sum_{n=-\infty}^{\infty}(...)-\sum_{n=0}\frac{z^2}{m^2}\right)+1-S=\\
&= \frac{1}{z^2}\left(\frac{3}{\pi^2}\sum_{m\neq 0}\sum_{n=-\infty}^{\infty}(...)\right)+1-\underbrace{\frac{3}{\pi^2}\sum_{m\neq 0}\frac{1}{m^2}}_{=1}-S=z^{-2}E_2\left(\frac{-1}{z}\right)-S \qquad (8)
\end{aligned}$$

$E_2\left(\frac{-1}{z}\right)$ exists because $\Im(\frac{-1}{z})>0$ and the existence of S is analog to the existence of $E_2(z)$.This justifies writing:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\sum_{m\neq 0}a_{m,n} &= \lim_{N\to\infty}\sum_{n=-N+1}^{N}\sum_{m\neq 0}a_{m,n}\\
&= \lim_{N\to\infty}\sum_{m\neq 0}\sum_{n=-N+1}^{N}a_{m,n}=\lim_{N\to\infty}\sum_{m\neq 0}\left(\frac{1}{mz-N}-\frac{1}{mz+N}\right)\\
&= \lim_{N\to\infty}\frac{2}{z}\sum_{m=1}^{\infty}\left(\frac{1}{-N/z+m}+\frac{1}{-N/z-m}\right)\\
&= \lim_{N\to\infty}\frac{2}{z}\left(\pi\cot\left(\frac{-\pi N}{z}\right)+\frac{z}{N}\right)\\
&= \frac{2\pi}{z}\lim_{N\to\infty}i\frac{e^{-2\pi iN/z}+1}{e^{-2\pi iN/z}-1}=-\frac{2\pi i}{z}
\end{aligned}$$

If we put this result in (8), we have shown Lemma 1.3 □

Definition 1.4.

$$\eta(z):=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

Satz 1.5. *$\eta(z)$ is absolutely convergent and*

$$\eta(\frac{-1}{z}) = \sqrt{(z/i)}\eta(z) \tag{9}$$

(We will use $\sqrt{z}$ with $0 \leq \Re(\sqrt{z})$)

Proof

Since $|q| < 1$ we have normal convergence at $\mathbb{H} \Rightarrow \eta(z)$ is holomorphic.
We will show that the logarithmic derivatives of the left and right side of (9) are equal. Then (9)must hold up to a multiplicative constant. With z=i we get that the constant must be 1. Remember $q' = 2\pi i q$

$$\frac{\eta(z)'}{\eta(z)} = \frac{2\pi i}{24}\left(1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right) = \frac{2\pi i}{24}\left(1 - 24\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} nq^{n(k+1)}\right) =^{(7)} \frac{2\pi i}{24}E_2(z) \tag{10}$$

$$\begin{aligned}\Rightarrow \frac{\left(\sqrt{(z/i)}\eta(z)\right)'}{\sqrt{(z/i)}\eta(z)} &= \frac{1}{2z} + \frac{\eta\prime(z)}{\eta(z)} =^{(10)} \frac{2\pi i}{24}\left(E_2(z) + \frac{12}{2\pi i z}\right) \\ &= \frac{2\pi i}{24}\left(z^{-2}E_2(-1/z)\right) =^{(10)} \frac{\eta'(-1/z)}{\eta(-1/z)}z^{-2} = \frac{(\eta(-1/z))'}{\eta(-1/z)}\end{aligned}$$

□

Korollar 1.6. *$\eta^{24}(z)$ is a cusp form of weight 12*

Proof

$\eta^{24}(z+1) = \eta^{24}(z)$ and $a_0 = 0$ of the fourier expansion is clear.

$$\eta^{24}(-1/z) =^{(9)} z^{12}\eta^{24}(z)$$

□

Proof of Satz 1.1

But $\dim(S_{12}) = 1 \Rightarrow \eta^{24}(z) = c\Delta(z), \texttt{ for some } c \in \mathbb{C}$.
Now we can compare the fourier expansion an get $c = (2\pi)^{-12}$

2 q-Expansion of η

Theorem 2.1.

$$\eta(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24} \tag{11}$$

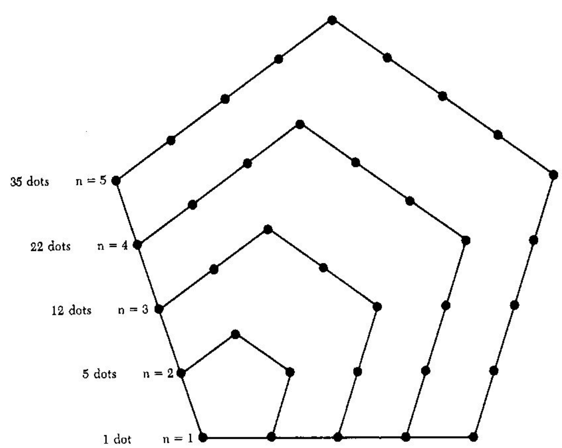
Theorem 2.1 follows directly from

Theorem 2.2. *Eulers pentogonal numbers theorem*

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2}$$

Remark

From the following figure, on sees that the number of dots on the nth pentagon is $n(3n-1)/2$. Therefore the numbers 1, 5, 12,... are called pentagonal numbers.



First Proof with elliptic functions

This proof can be found in M.Koecher, A.Krieg, Elliptische Funktionen und Modulformen.

Lemma 2.3. *Jacobi's triple product We set* $\Theta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau/2 + 2\pi i n z}$ *with* $z \in \mathbb{C}, \tau \in \mathbb{H}$. *Then*

$$\Theta(z,\tau) = \prod_{m=1}^{\infty}(1-e^{2\pi i m\tau})(1+e^{(2m-1)\pi i\tau+2\pi i z})(1+e^{(2m-1)\pi i\tau-2\pi i z})$$

Proof of theorem 2.1
We replace τ by 3τ and z by $\frac{\tau+1}{2}$

$$\begin{aligned}\Theta(\frac{\tau+1}{2},3\tau) &= \prod_{m=1}^{\infty}(1-e^{2\pi i m 3\tau})(1+e^{(2m-1)\pi i 3\tau+2\pi i(\tau+1)/2})(1+e^{(2m-1)\pi i 3\tau-2\pi i(\tau+1)/2}) \\ &= \prod_{m=1}^{\infty}(1-e^{3(2\pi i m\tau)})(1+e^{(3m-1)2\pi i\tau+\pi i})(1+e^{(3m-2)2\pi i\tau-\pi i}) \\ &= \prod_{m=1}^{\infty}(1-q^{3m})(1-q^{(3m-1)})(1-q^{(3m-2)}) = \prod_{m=1}^{\infty}(1-q^m)\end{aligned}$$

□

Idea of the proof of lemma 2.3
We set $g(z,\tau) = \prod_{m=1}^{\infty}(1-e^{2\pi i m\tau})(1+e^{(2m-1)\pi i\tau+2\pi i z})(1+e^{(2m-1)\pi i\tau-2\pi i z})$ and fix τ.
One can show, that $\frac{\Theta(z,\tau)}{g(z,\tau)}$ is an elliptic function and that $\Theta(z,\tau)$ and $g(z,\tau)$ have the same zeros. By Liouville $\frac{\Theta(z,\tau)}{g(z,\tau)}$ is constant.
Now let τ be arbitrary. Then there exists a holomorphic function φ such that $\varphi(q) = \frac{\Theta(z,\tau)}{g(z,\tau)}$. By a direct calculation, one gets $\varphi(q) = \varphi(q^4)$. One also easy sees, that one can extend $\varphi(q)$ to $q = 0$ holomorphicly by $\varphi(0) = 1$. I follows by the identity theorem, that $\varphi(q) \equiv 1$. □

Second Proof with modular forms
This proof can be found in Freitag, Busam, Funktionentheorie 1
We will show

Lemma 2.4.

$$(2\pi)^{-12}\Delta(z) = q\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}\right)^{24}$$

Proof of theorem 2.1
Using the Theorem 1.1 and the definition of η, one gets

$$\left(\prod_{m=1}^{\infty}(1-q^m)\right)^{24} = \left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}\right)^{24}$$

We set U to be the preimage of the set $z \in \mathbb{C}\setminus\mathbb{R}^-$ of $(\prod_{m=1}^{\infty}(1-q^m))^{24}$. On U we have

$$\epsilon\prod_{m=1}^{\infty}(1-q^m) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2},\ \epsilon^{24}=1$$

and therefore on the whole $\mathbb{H}$. Comparing the fourier coefficient of q^0, one gets $\epsilon = 1$. □

Idea of the Proof of 2.4

We will show that $f(z) := q\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}\right)^{24}$ is a cusp form of weight 12.
It is clear, that f has a 0 at infinity. Since $3n^2+n$ is always even, one gets $f(z+1) = f(z)$. For the last condition, we have to work a little bit more. We see $f(z) = \Theta(3z, \frac{z+1}{2})$ and $f(\frac{-1}{z}) = \Theta(\frac{-3}{z}, \frac{1}{2} - \frac{1}{2z})$. We need the θ-transformation formula:

$$\Theta\left(\frac{-1}{z}, \tau\right) = \sqrt{\frac{z}{i}} \sum_{n=-\infty}^{\infty} e^{\pi i (n+\tau)^2 z}$$

By a direct calculation one gets

$$f\left(\frac{-1}{z}, \tau\right) = \sqrt{\frac{z}{i}} e^{\left(\frac{\pi i z}{12} + \frac{\pi i}{12z}\right)} f(z)$$

□

Third Proof with partition functions

Let $n, m \in \mathbb{N}$. A partition of n is sequence $(\lambda_i)_{i\in\mathbb{N}}$ with $\lambda_i \geq \lambda_{i+1}$ and $\sum_i \lambda_i = n$. Let $P(n)$ be the number of partitions of n.

Lemma 2.5.

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} P(n) q^n \tag{12}$$

Proof Using the geometric series □

Remark $P(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{\sqrt{n-1/24}}$, $n \to \infty$. This expression was first obtained by G. H. Hardy and Ramanujan in 1918 and independently by J. V. Uspensky in 1920.

We set $\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=0}^{\infty} a_n q^n$. Using Lemma 12 we get

$$\left(\sum_{n=0}^{\infty} P(n) q^n\right)\left(\sum_{n=0}^{\infty} a_n q^n\right) = 1$$

Comparing coefficients we have

$$a_0 P(0) = a_0 = 1 \tag{13}$$

$$\sum_{k=0}^{m} P(k) a_k = 0, \ \forall m > 0 \tag{14}$$

From this two equations one sees that the a_n are uniquely determined. From Theorem 2.2 one can extract possible values for the a_n's.We only have to show, that they fulfil (13) and (14) and we are done.
We reformulating (14) and get an equation between to different families of partitions. As last thing we construct an bijection between those families.

3 Congruences for the coefficients of the j function

pro memoria:

$$j(z) = \frac{(12g_2(z))^3}{\Delta(z)} = \frac{(720G_4(z))^3}{\Delta(z)}$$

Proposition 3.1. *The j function has fourier expansion*

$$q^{-1} + \sum_{m=0}^{\infty} j_m q^m \quad j_m \in \mathbb{Z} \ \forall m$$

Proof We use the following Lemma

Lemma 3.2. *Let f and g be convergent power series for $|q| < 1$*

$$f(q) = \sum_{n=0}^{\infty} a_n q^n, \ g(q) = \sum_{n=0}^{\infty} b_n q^n \quad a_n, b_n \in \mathbb{Z} \ \forall n$$

with $b_0 = 1$ and $g(q) \neq 0$ for all $|q| < 1$. Then $\frac{f}{g}$ is also a convergent power series for $|q| < 1$ with integer coefficients

Proof Since $\frac{f}{g}$ is a holomorphic function, it has a convergent power series for $|q| < 1$ with coefficients $c_n \in \mathbb{C}$. We get

$$\left(\sum_{n=0}^{\infty} c_n q^n\right)\left(\sum_{n=0}^{\infty} b_n q^n\right) = \left(\sum_{n=0}^{\infty} a_n q^n\right)$$

Comparing coefficients we get

$$c_0 = a_0, \ c_m = a_m - \sum_{n=0}^{m-1} c_n b_{m-n}$$

□

Now there are certain interesting properties for the coefficents

Theorem 3.3.

- $j_m \sim \frac{e^{4\pi\sqrt{m}}}{\sqrt{2}m^{3/4}}$ *for* $m \to \infty$
- $(m+1)j_m \equiv 0 \ (mod\ 24) \ \forall m \geq 1$
- *For* $m \equiv 0 \ (mod\ 2^a 3^b 5^c 7^d)$ *we have*
  $j_m \equiv 0 \ (mod\ 2^{4a+8} 3^{2b+3} 5^{c+1} 7^d)$

The first property is due H.Petersen and H. Rademacher.

The second and the third is due D.H.Lehmer.

We will not proof this Theorem, but give an idea of the proof of the following theorem

Theorem 3.4.

- $j(2n) \equiv 0 \ (mod \ 2^{11})$
- $j(2n) \equiv 0 \ (mod \ 3^{5})$
- $j(2n) \equiv 0 \ (mod \ 5^{2})$
- $j(2n) \equiv 0 \ (mod \ 7)$

Short idea of the Proof

We take a look at the case 5^2. We set

$$j_5(z) = \sum_{n=1}^{\infty} j(5m)q^m$$

We will try to find an function Φ such that

$$j_5(z) = 25\left(a_1\Phi(z) + a_2\Phi^2(z) + \cdots + a_k\Phi^k(z)\right)$$

where Φ has fourier expansion with integer coefficients.
The first idea is to take $\Phi = j$, since we know that all modular functions of weight 0 are rational functions of j. The problem is that $j_5(z)$ is not anymore an modular function. But we will find a subgroup of Γ, such that $j_5(z)$ will be modular for this subgroup and construct a function Φ which will play the same role as the j function for Γ

Preparations

Definition 3.5. Let p be a prim. We set

$$\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | c \equiv 0 \ (mod \ p) \right\}$$

It ist easy to verify that $\Gamma_0(p)$ is a subgroup of Γ and that $T \in \Gamma_0(p)$ and $S \notin \Gamma_0(p)$

Definition 3.6. A function f is called an automorphic function for $\Gamma_0(p)$ if the following conditions hold:

- f is meromorphic in $\mathbb{H}$
- $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma_0(p)$
- f has at at maximum a pole at infinity

There is an way to construct automorphic function for $\Gamma_0(p)$.

Proposition 3.7. *Let f be an automorphic function for Γ and p be a prim. We set*

$$f_p(z) := \frac{1}{p}\sum_{d=0}^{p-1} f\left(\frac{z+d}{p}\right)$$

Then $f_p(z)$ is autmorphic under $\Gamma_0(p)$. If f has the fourier expansion

$$f(z) = \sum_{n=-m}^{\infty} a(n)q^n$$

then $f_p(z)$ has the fourier expansion

$$f_p(z) = \sum_{n=-[m/p]}^{\infty} a(np)q^n$$

Proof

$$pf_p(z) = \sum_{d=0}^{p-1}\sum_{n=-m}^{\infty} a(n)e^{2\pi in(z+d)/p} = \sum_{n=-m}^{\infty} a(n)e^{2\pi inz/p}\sum_{d=0}^{p-1} e^{2\pi ind/p}$$

But

$$\sum_{d=0}^{p-1} e^{2\pi ind/p} = \begin{cases} 0, & \text{p don't divides n} \\ p, & \text{else} \end{cases}$$

So we have shown the second part and that f has at maximum a pole at infinity. It is also clear, that $f_p(z)$ is meromorphic.
For the proof of the modularity we need the following lemma

Lemma 3.8. *If $\gamma \in \Gamma_0(p)$ and if $1 \leq d \leq p-1$, let $T_d(z) = \frac{z+\lambda}{p}$. Then there exists a integer μ, $1 \leq \mu \leq p-1$ and a transformation $\gamma_\mu \in \Gamma_0(p)$ s.t.*

$$T_d\gamma = \gamma_\mu T_\mu$$

Moreover, as d runs through a complete residue system modulo p, so does μ.

Proof: direct calcutaion. □

We get

$$\begin{aligned} f_p(\gamma z) &= \frac{1}{p}\sum_{d=0}^{p-1} f\left(\frac{\gamma z + d}{p}\right) = \frac{1}{p}\sum_{d=0}^{p-1} f(T_d(\gamma z)) = \frac{1}{p}\sum_{d=0}^{p-1} f(\gamma_\mu T_\mu(z)) \\ &= \frac{1}{p}\sum_{d=0}^{p-1} f(T_\mu(z)) = f_p(z) \end{aligned}$$

□

Applying a lemma similar to Lemma 3.8 to $f(Sz)$, we get

Proposition 3.9. *If f is an automorpic under Γ an p a prim, then*

$$pf_p(-1/z) = pf_p(z) + f(pz) - f(z/p)$$

Applying this to the fourier expansion of j and j_p one finds

Proposition 3.10.

$$pj_p(\frac{-1}{pz}) = q^{-p^2} - q^{-1} + I(q) \tag{15}$$

where $I(q)$ is a powers series with integer coefficents.

The construction of Φ

Lemma 3.11. *Let p be a prime. We set*

$$\phi(z) = \frac{\Delta(pz)}{\Delta(z)}$$

Then ϕ ist automorphic under $\Gamma_0(p)$ an has a fourier expansion of the form

$$\phi(z) = q^{p-1}\left(1 + \sum_{n=1}^{\infty} b_n q^n\right)$$

with b_n integers. We have also

$$\phi(-1/qz) = \frac{1}{q^{12}\phi(z)} \tag{16}$$

Proof: Theorem 1.1, Lemma 3.2 and calculating.

Remark $\Gamma_0(p)$ is important for the Proof.

If we set $\alpha := 1/(p-1)$ one gets

$$\phi^\alpha = \epsilon q \left(1 + \sum_{n=1}^{\infty} b_n q^n\right)^\alpha \quad , \epsilon^{p-1} = 1$$

ϕ^α has a simple zero at infinity, but now the coefficients of ϕ^α are a priori not integers. One can show that $\epsilon = 1$. Using the Definition of ϕ and Theorem 1.1 one gets

$$\phi(z) = \frac{\Delta(pz)}{\Delta(z)} = q^{n-1} \frac{\prod_{n=1}^{\infty}(1-q^{pn})^{24}}{\prod_{n=1}^{\infty}(1-q^n)^{24}} = q^{p-1}\left(1 + \sum_{n=1}^{\infty} d_n q^n\right)^{24}$$

The fourier coefficients of ϕ^α are certainly integers if α divides 24. This occurs if p=2,3,5,7,13. Therefore we set for those p

$$\Phi(z) := \phi^\alpha$$

We collect everything and use Lemma 3.2

Proposition 3.12. *For p=2,3,5,7,13 $\Phi(z)$ is an automorphic form for $\Gamma_0(p)$ an has a fourier expansion of the form*

$$\frac{1}{\Phi(z)} = \frac{1}{q} + I(q)$$

Where $I(q)$ is a power series with integer coefficients.

Theorem 3.13. *For p=2,3,5,7,13 we set r=24/(p-1). Then there exists $a_1, ..., a_{p^2}$ such that*

$$j_p(z) = p^{r/2-1}\left(a_1\Phi(z) + a_2\Phi^2(z) + ... + a_{p^2}\Phi^{p^2}\right) + j(0)$$

Proof

Using Equation 16 on gets

$$\Psi(z) := p^{r/2}\Phi(\frac{-1}{pz}) = \frac{1}{\Phi(z)} = q^{-1} + I(q)$$

We see with Equation 15 that

$$pj_p(\frac{-1}{pz}) - (\Psi(z))^{p^2}$$

has a pole of order $\leq p^2 - 1$ at 0 and integer coefficients. After finite steps one gets

$$f(\frac{-1}{pz}) = pj_p(\frac{-1}{pz}) - (\Psi(z))^{p^2} - b_1(\Psi(z))^{p^2-1} - ... - b_{p^2-1}\Psi(z)$$

with no pole at 0 and integer coefficients b_i

Replacing $-1/pz$ by z, we have

$$f(z) = pj_p(z) - (p^{r/2}\Phi(z))^{p^2} - b_1(p^{r/2}\Phi(z))^{p^2-1} - ... - b_{p^2-1}p^{r/2}\Phi(z)$$

f is automorphic and analytic at each point in $\mathbb{H}$. We also have $\lim_{z\to\infty} |f(z)|$ is bounded by construction. $\square$

Proof of Theorem 3.4

Watch the Tabular

p	2	3	5	7	13
r/2-1	11	5	2	1	0

$\square$

Remark It is known, that $j(13)$ is not divisible by 13.