On the singularity formation for the energy critical wave map to the 2-sphere

Pierre Raphaël, IMT, Toulouse, France

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One of the simplest model of geometric wave equations are wave maps which are maps from Minkowski space ($\mathbb{R} \times \mathbb{R}^m$) into a n-dimensional Riemanian manifold (M, g), and which are critical points of the natural scalar Lagrangian density associated to the Minkowski metric. The question of the global existence or the possible finite time breakdown of the associated flow has attracted a considerable attention and turns out to be a delicate issue deeply related to the geometry of the target manifold.

I shall restrict in this talk onto the case of 2 dimensional wave maps with target the 2-sphere which is an *energy critical problem.* Under the additional assumption of equivariant symmetry, the flow reduces to the following semilinear wave equation for the Euler angle

$$\partial_{tt}\phi - \partial_{rr}\phi - \frac{\partial_r\phi}{r} + \frac{k^2\sin(2\phi)}{2r^2} = 0,$$

where the integer $k \geq 1$ is the homotopy number. The existence of blow up solutions for this problem has been obtained very recently in two breathrough works corresponding to two different settings. On the one hand, for a high homotopy number $k \geq 4$ -and hence a high energy level-, Rodnianski and Sterbenz have proved the existence of a *stable* finite time blow up dynamic for which a -non sharp- upper bound on blow up rate is established. On the other hand, near the fundamental energy level $k = 1$, a completely different blow up regime of *slow* blow up solutions with different blow up rates has been exhibited by Krieger, Schlag, Tataru, which is expected to describe the generically *unstable* blow up manifold.

In this talk, I will present a unified framework to derive the existence of stable -in the equivariant class- finite time blow up dynamics for all homotopy number $k \geq 1$, including in particular the fundamental level $k = 1$. Moreover, I will give *sharp* asymptotics on the singularity formation including exact blow up rates and a proof of the *quantization* of the energy focused by the singularity.

This is joint work with Igor Rodnianski (Princeton) and Jacob Sterbenz (UCSD).