Modelling of Lévy term structures

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Term structure models in the context of credit risk which have been studied in the literature so far are typically models driven by a Brownian motion, or are standard jump diffusions. See Zhou (1997), Schönbucher (1998), or Duffie and Singleton (1999) for examples, and the surveys on credit risk models in Lando (1997), Ammann (1999), and Schönbucher (2000).
A new approach to credit risk based on the term structure methodology of Heath, Jarrow, and Morton (1992) was introduced by Bielecki and Rutkowski (1999, 2000). The Bielecki-Rutkowski model takes the information on rating migration and on credit spreads into account and yields an arbitrage-free model of defaultable bonds. We follow this approach to construct an intensity-based credit risk framework for term structure models driven by Lévy processes.
We start with the default free instantaneous forward rate $f(t,T)$ given by

$$df(t,T) \;=\; \partial_2 A(t,T)dt - \partial_2 \Sigma(t,T)^\top dL_t \tag{1}$$

where $A(t,T)$ and $\Sigma(t,T)$ are processes satisfying certain smoothness conditions and (L_t) is a d-dimensional Lévy process which in the canonical decomposition can be written as

$$L_t \;=\; bt + cW_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \nu^L)(ds,dx).$$

The corresponding price of a default-free bond is then given by

$$B(t,T) \;=\; B(0,T)\exp\left(\int_0^t (r(s) - A(s,T))\;ds + \int_0^t \Sigma(s,T)^\top c\,dW_s + \int_0^t \int_{\mathbb{R}^d} \Sigma(s,T)^\top x(\mu^L - \nu^L)(ds,dx)\right). \tag{2}$$

Now assume that an internal rating system $\mathcal{K} = \{1,\ldots,K\}$ is given. Class 1 corresponds to the best possible rating following default-freeness – which is denoted AAA in the Standard & Poor's rating – class K corresponds to default. The instantaneous forward rate for class $i \in \{1, \dots, K-1\}$ is assumed to satisfy the equation

$$(3) \qquad dg_i(t,T) \;=\; \partial_2 A_i(t,T)\,dt - \partial_2 \Sigma_i(t,T)^\top\, dL_t^{(i)},$$

where $(L_t^{(i)})$ is given by $L_t^{(i)} = b_i t + c_i W_t + \int_0^t \int_{\mathbb{R}^d} p_i x(\mu^L - \nu^L)\,(ds,dx)$. In order to ensure that risky corporate bonds have higher forward rates than less risky bonds, we assume

$$(4) \qquad g_{K-1}(t,T) \;>\; g_{K-2}(t,T) > \cdots > g_1(t,T) > f(t,T).$$

The dynamics of the conditional bond price based on the forward rate g_i can then be derived in the form

$$\begin{aligned} dD_i(t,T) \;=\; & D_i(t-,T)\Big(\big(a_i(t,T) + g_i(t,t)\big)\,dt + \int_{\mathbb{R}^d} \Sigma_i(t,T)^\top p_i x \left(\mu^L - \nu^L\right)(dt,dx) \\ (5) \qquad & + \Sigma_i(t,T)^\top c_i\, dW_t + \int_{\mathbb{R}^d}\Big(e^{\Sigma_i(t,T)^\top p_i x} - 1 - \Sigma_i(t,T)^\top p_i x\Big)\,\mu^L(dt,dx)\Big), \end{aligned}$$

where $a_i(t,T) = \frac{1}{2}\left|\Sigma_i(t,T)^\top c_i\right|^2 - A_i(t,T)$. The
credit migration process is modelled by a conditional Markov process C on the space of rating classes $\mathcal{K}$. We define $H_i(t) = 1\!\!1_{\{s\geq 0 | C_s = i\}}(t)$ and write for $i \neq j$, $H_{ij}(t)$ for the number of transitions from rating i to rating j in the time interval $[0,t]$. Then the time t price $D_C(t,T)$ of a defaultable bond maturing at time T, which is currently rated at C_t, equals

$$(6) \qquad D_C(t,T) \;=\; B(t,T)\sum_{i=1}^{K-1}\left(H_i(t)\exp\left(-\int_t^T \gamma_i(t,u)du\right) + \delta_i H_{i,K}(t)\right)$$

where $\gamma_i(t,u) = g_i(t,u) - f(t,u)$ is the i-th forward credit spread and $\delta_i \in [0,1)$ the corresponding recovery rate which is assumed to be constant. This defaultable bond price can also be expressed in terms of a risk-neutral valuation formula, i.e. as an expectation with respect to a martingale measure. Further extensions of the model to include reorganization of firms and multiple defaults are considered. This is joint work with Fehmi Özkan.

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